7. Problemset Relativistic Quantum Field Theory
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7.1 Symmetry Generators

Consider \( n \) real scalar fields \( \{ \phi_i \}_{i=1,...,n} \)

\[
\phi_i(x) = \int \tilde{d}k \left( a_i(k)e^{-ikx} + a_i^\dagger(k)e^{ikx} \right) \tag{1a}
\]

\[
\pi_i(x) = -i \int \tilde{d}k k_0 \left( a_i(k)e^{-ikx} - a_i^\dagger(k)e^{ikx} \right) \tag{1b}
\]

and the charges

\[
Q_a = i \sum_{i,j=1}^{n} \pi_i T_{ij}^a \phi_j \tag{2}
\]

where the \( n \times n \) matrices form a Lie algebra

\[
[T_a, T_b] = i \sum_c f_{abc} T_c . \tag{3}
\]

1. Compute the commutation relations \([Q_a, Q_b]\) using the canonical commutation relations

\[
[\phi_i(t, \vec{x}), \pi_j(t, \vec{y})] = i \delta_{ij} \delta^3(\vec{x} - \vec{y}) \tag{4a}
\]

\[
[\phi_i(t, \vec{x}), \phi_j(t, \vec{y})] = [\pi_i(t, \vec{x}), \pi_j(t, \vec{y})] = 0 . \tag{4b}
\]

2. Express the \( \{ Q_a \} \) in terms of the creation and annihilation operators \( a_i \) and \( a_i^\dagger \).

3. When do you have to normal order the \( \{ Q_a \} \)?

4. Can you normal order the \( \{ Q_a \} \) without spoiling their commutation relations?
7.2 Spin-1/2 Energy Momentum Tensor

In the lecture, it was argued that the Lagrangian densities

\[ L_1 = \bar{\psi} (i\gamma^\mu \partial_\mu - m) \psi \] (5a)

\[ L_2 = \frac{i}{2} \left( \bar{\psi} \gamma^\mu (\partial_\mu \psi) - (\partial_\mu \bar{\psi}) \gamma^\mu \psi \right) - m \bar{\psi} \psi \] (5b)

both lead to the free Dirac equations

\[ (i\gamma^\mu \partial_\mu - m) \psi = 0 \] (6a)

\[ \bar{\psi} \left( i\gamma^\mu \partial_\mu + m \right) \psi = 0 \] (6b)

if independent variations of \( \{ \psi_\alpha(x) \}_{\alpha=1,2,3,4} \) and \( \{ \psi^*_\alpha(x) \}_{\alpha=1,2,3,4} \) and are used in the Euler-Lagrange-Equations

\[ \partial_\mu \left( \frac{\partial L}{\partial \psi^{\alpha}} - \frac{\partial L}{\partial \psi^{\alpha}} \right) = 0 \] (7a)

\[ \partial_\mu \left( \frac{\partial L}{\partial \psi^{\alpha}} - \frac{\partial L}{\partial \psi^{\alpha}} \right) = 0 \]. (7b)

1. Show that this is true.

2. Compute the energy momentum tensors \( \Theta^1_{\mu\nu} \) and \( \Theta^2_{\mu\nu} \) using \( L_1 \) and \( L_2 \), respectively.

3. Do they agree?

4. Are they symmetric?

5. Compute the Hamiltonians

\[ H_i = \int d^3x \Theta^{00}_i \] (8)

for \( i = 1, 2 \).

6. Express the Hamiltonian(s) for the Dirac field

\[ \psi^\alpha(x) = \sum_{\sigma=\uparrow,\downarrow} \int d^3p \left( u^\alpha_\sigma(p) c_\sigma(p)e^{-ixp} + v^\dagger_\sigma(p)d_{\sigma(p)}e^{ixp} \right) \] (9)

in terms of the creation and annihilation operators. Do you have to apply normal ordering?
7.3 Charged Scalar Field

In the lecture, it was argued that the Lagrangian densities

\[
\mathcal{L}_1 = \sum_{i=1}^{2} \left( \frac{1}{2} \partial_\mu \phi_i \partial^\mu \phi_i - \frac{m^2}{2} \phi_i^2 \right) - \frac{\lambda}{8} \left( \phi_1^2 + \phi_2^2 \right)^2 \tag{10a}
\]

\[
\mathcal{L}_2 = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi - \frac{\lambda}{2} (\phi^* \phi)^2 \tag{10b}
\]

lead to equivalent equations of motion, provided that

\[
\phi(x) = \frac{1}{\sqrt{2}} (\phi_1(x) + i\phi_2(x)) \tag{11}
\]

and \(\phi^*(x)\) are varied independently.

1. Derive the equations of motion for \(\phi_1\) and \(\phi_2\) from \(\mathcal{L}_1\).
2. Derive the equations of motion for \(\phi\) and \(\phi^*\) from \(\mathcal{L}_2\).
3. Verify that they are equivalent.
4. Derive conjugate momenta, the Hamiltonians \(H_1\) and \(H_2\) and the canonical equations of motion resulting from \(\mathcal{L}_1\) and \(\mathcal{L}_2\) respectively.
5. Verify that they are equivalent.