7. Problemset Relativistic Quantum Field Theory
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7.1 Symmetry Generators

Consider n real scalar fields $\{\phi_i\}_{i=1,\ldots,n}$

$$\phi_i(x) = \int \widetilde{\mathrm{d}k} \left(a_i(k)\mathrm{e}^{-\mathrm{i}kx} + a_i^\dagger(k)\mathrm{e}^{\mathrm{i}kx} \right) \tag{1a}$$

$$\pi_i(x) = -\mathrm{i} \int \widetilde{\mathrm{d}k}\, k_0 \left(a_i(k)\mathrm{e}^{-\mathrm{i}kx} - a_i^\dagger(k)\mathrm{e}^{\mathrm{i}kx} \right) \tag{1b}$$

and the charges

$$Q_a = \mathrm{i} \sum_{i,j=1}^{n} \pi_i T_a^{ij} \phi_j \tag{2}$$

where the $n \times n$ matrices form a Lie algebra

$$[T_a, T_b] = \mathrm{i} \sum_c f_{abc} T_c \,. \tag{3}$$

1. Compute the commutation relations $[Q_a, Q_b]$ using the canonical commutation relations

$$[\phi_i(t,\vec{x}), \pi_j(t,\vec{y})] = \mathrm{i}\delta_{ij}\delta^3(\vec{x} - \vec{y}) \tag{4a}$$

$$[\phi_i(t,\vec{x}), \phi_j(t,\vec{y})] = [\pi_i(t,\vec{x}), \pi_j(t,\vec{y})] = 0 \,. \tag{4b}$$

2. Express the $\{Q_a\}$ in terms of the creation and annihilation operators a_i and $a_i^\dagger$.

3. When do you have to normal order the $\{Q_a\}$?

4. Can you normal order the $\{Q_a\}$ without spoiling their commutation relations?

7.2 Spin-1/2 Energy Momentum Tensor

In the lecture, it was argued that the Lagrangian densities

$$\mathcal{L}_1 = \bar{\psi}\,(\mathrm{i}\partial\!\!\!/ - m)\,\psi \tag{5a}$$

$$\mathcal{L}_2 = \frac{\mathrm{i}}{2}\left(\bar{\psi}\gamma^\mu(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\gamma^\mu\psi\right) - m\bar{\psi}\psi \tag{5b}$$

both lead to the free Dirac equations

$$(\mathrm{i}\partial\!\!\!/ - m)\,\psi = 0 \tag{6a}$$

$$\bar{\psi}\left(\mathrm{i}\overleftarrow{\partial\!\!\!/} + m\right) = 0 \tag{6b}$$

if independent variations of $\{\psi_\alpha(x)\}_{\alpha=1,2,3,4}$ and $\{\psi^*_\alpha(x)\}_{\alpha=1,2,3,4}$ and are used in the Euler-Lagrange-Equations

$$\partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\psi^\alpha} - \frac{\partial\mathcal{L}}{\partial\psi^\alpha} = 0 \tag{7a}$$

$$\partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\psi^*_\alpha} - \frac{\partial\mathcal{L}}{\partial\psi^*_\alpha} = 0\,. \tag{7b}$$

1. Show that this is true.
2. Compute the energy momentum tensors $\Theta_1^{\mu\nu}$ and $\Theta_2^{\mu\nu}$ using $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively.
3. Do they agree?
4. Are they symmetric?
5. Compute the Hamiltonians

   $$H_i = \int \mathrm{d}^3\vec{x}\,\Theta_i^{00} \tag{8}$$

   for $i = 1, 2$.
6. Express the Hamiltonian(s) for the Dirac field

   $$\psi^\alpha(x) = \sum_{\sigma=\uparrow,\downarrow}\int\widetilde{\mathrm{d}p}\,\left(u^\alpha_\sigma(p)c_\sigma(p)\mathrm{e}^{-\mathrm{i}xp} + v^\alpha_\sigma(p)d^\dagger_\sigma(p)\mathrm{e}^{\mathrm{i}xp}\right) \tag{9}$$

   in terms of the creation and annihilation operators. Do you have to apply normal ordering?

7.3 Charged Scalar Field

In the lecture, it was argued that the Lagrangian densities

$$\mathcal{L}_1 = \sum_{i=1}^{2} \left(\frac{1}{2} \partial_\mu \phi_i \partial^\mu \phi_i - \frac{m^2}{2} \phi_i^2 \right) - \frac{\lambda}{8} \left(\phi_1^2 + \phi_2^2 \right)^2 \tag{10a}$$

$$\mathcal{L}_2 = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi - \frac{\lambda}{2} \left(\phi^* \phi \right)^2 \tag{10b}$$

lead to equivalent equations of motion, provided that

$$\phi(x) = \frac{1}{\sqrt{2}} \left(\phi_1(x) + \mathrm{i} \phi_2(x) \right) \tag{11}$$

and $\phi^*(x)$ are varied independently.

1. Derive the equations of motion for ϕ_1 and ϕ_2 from $\mathcal{L}_1$.
2. Derive the equations of motion for ϕ and ϕ^* from $\mathcal{L}_2$.
3. Verify that they are equivalent.
4. Derive conjugate momenta, the Hamiltonians H_1 and H_2 and the canonical equations of motion resulting from $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively.
5. Verify that they are equivalent.