6. Problemset Relativistic Quantum Field Theory
November 29, 2017

Dirac meets Majorana

We will study the Dirac algebra relations

$$[\gamma^\mu, \gamma^\nu]_+ = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} \cdot \mathbf{1} \tag{1}$$

and their consequences.

6.1 Poincaré Algebra

Show that

$$[\sigma^{\mu\nu}, \gamma^\rho]_- = 2\mathrm{i} \left(\gamma^\mu g^{\nu\rho} - \gamma^\nu g^{\mu\rho}\right) \tag{2a}$$
$$[\sigma^{\mu\nu}, \sigma^{\rho\sigma}]_- = 2\mathrm{i} \left(g^{\mu\rho}\sigma^{\nu\sigma} - g^{\mu\sigma}\sigma^{\nu\rho} - g^{\nu\rho}\sigma^{\mu\sigma} + g^{\nu\sigma}\sigma^{\mu\rho}\right) \tag{2b}$$

using *only* the Dirac algebra relations (1) and the definition

$$\sigma^{\mu\nu} = \frac{\mathrm{i}}{2} [\gamma^\mu, \gamma^\nu]_- = \frac{\mathrm{i}}{2} \left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right) . \tag{3}$$

6.2 Contraction Identities

Prove the contraction Identities for Dirac matrices

$$\gamma^\mu\gamma_\mu = 4 \cdot \mathbf{1} \tag{4a}$$
$$\gamma^\mu \not{a} \gamma_\mu = -2\not{a} \tag{4b}$$
$$\gamma^\mu \not{a}\not{b} \gamma_\mu = 4ab \cdot \mathbf{1} \tag{4c}$$
$$\gamma^\mu \not{a}\not{b}\not{c} \gamma_\mu = -2\not{c}\not{b}\not{a} \tag{4d}$$

and compute

$$\gamma^\mu \not{a}\not{b}\not{c}\not{d} \gamma_\mu \tag{5a}$$
$$\gamma^\mu \sigma^{\kappa\lambda} \gamma_\mu \tag{5b}$$
$$\gamma^\mu \sigma^{\kappa\lambda} \not{a} \gamma_\mu \tag{5c}$$

using *only* the Dirac algebra relations (1).

6.3 Majorana Representation

1. Show that the matrices

$$\gamma^0 = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix} \tag{6a}$$

$$\gamma^1 = \begin{pmatrix} \mathrm{i}\sigma_3 & 0 \\ 0 & \mathrm{i}\sigma_3 \end{pmatrix} \tag{6b}$$

$$\gamma^2 = \begin{pmatrix} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{pmatrix} \tag{6c}$$

$$\gamma^3 = \begin{pmatrix} -\mathrm{i}\sigma_1 & 0 \\ 0 & -\mathrm{i}\sigma_1 \end{pmatrix} \tag{6d}$$

   form a realization of the Dirac algebra (1).

2. Compute the pseudoscalar element γ_5 of the Dirac Algebra.

3. Compute the generators of rotations $\sigma^{ij} = \mathrm{i}[\gamma^i, \gamma^j]_-/2$.

4. Compute the generators of boosts $\sigma^{0i} = \mathrm{i}[\gamma^0, \gamma^i]_-/2$.

5. Show that

$$C = \begin{pmatrix} 0 & -\mathrm{i}\sigma^2 \\ -\mathrm{i}\sigma^2 & 0 \end{pmatrix} \tag{7}$$

   is a charge conjugation matrix and show that $C \neq \mathrm{i}\gamma^2\gamma^0$ in this representation.

6. Can you identify a special feature of this representation? *Hint: look at the explicit form of the Dirac equation* $(\mathrm{i}\not\partial - m)\psi = 0$.