5. Problemset Relativistic Quantum Field Theory
November 22, 2017

Superderivations / Commutators / Wightman Functions

5.1 Super Leibniz Rule

For the evaluation of commutators, the Leibniz rule

$$[A, BC]_- = B[A, C]_- + [A, B]_- C \tag{1}$$

is often more productive than using the definition via the associative product

$$[A, BC]_- = ABC - BCA \tag{2}$$

that implies (1). It is possible to derive generalized Leibniz rules that mix commutators and anticommutators.

Consider a system of bosonic operators a_i and fermionic operators c_i with elementary commutation relations

$$[a_i, c_j]_- = 0 \tag{3a}$$
$$[a_i, a_j]_- = B_{ij}\,. \tag{3b}$$

for bosonic and mixed pairs of operators and elementary anticommutation relations

$$[c_i, c_j]_+ = F_{ij} \tag{3c}$$

for fermionic pairs of operators.

1. Express $[a_i, c_j c_k]_-$ elementary commutators.
2. Express $[c_i, a_j c_k]_+$ elementary anticommutators.
3. Express $[c_i, c_j c_k]_-$ elementary anticommutators.
4. Express $[c_i c_j, c_k c_l]_-$ elementary anticommutators.
5. Express $[a_i c_j, a_k c_l]_+$ elementary commutators and anticommutators.
6. Can you formulate a general rule for expressing $[A, BC]_\pm$ in terms of $[A, B]_\pm$ and $[A, C]_\pm$ depending on which of A, B, C are fermionic or bosonic?

5.2 Commutator Function

Consider the positive energy part of the commutator function

$$\mathrm{i}\Delta^{(+)}(x;m) = \int \widetilde{\mathrm{d}p}\, \mathrm{e}^{-\mathrm{i}xp} = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2p_0} \mathrm{e}^{-\mathrm{i}xp}\Big|_{p_0=\sqrt{\vec{p}^2+m^2}} . \tag{4}$$

1. Compute $\Delta^{(+)}(x;m)$ explicitely for $x^2 < 0$.
   (a) Can you take the limit $m \to 0$?
   (b) Discuss the limit $x^2 \to 0$.
2. Compute $\Delta^{(+)}(x;m)$ explicitely for $x^2 > 0$, depending on the sign of x_0.
   (a) Can you take the limit $m \to 0$?
   (b) Discuss the limit $x^2 \to 0$.
3. Compute
   $$\Delta(x) = \Delta^{(+)}(x) - \Delta^{(+)}(-x) . \tag{5}$$
   (a) Can you find a compact expression?
   (b) Can you take the limit $m \to 0$?
   (c) Discuss the limits $x^2 \to 0$ from above and below.

5.3 Wightman Functions

Consider the charged scalar field

$$\phi(x) = \int \widetilde{\mathrm{d}p}\, \left(a(p)\mathrm{e}^{-\mathrm{i}xp} + a^{c\dagger}(p)\mathrm{e}^{\mathrm{i}xp}\right) . \tag{6}$$

1. For which n and m do the Wightman functions
   $$W(x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_m) = \left\langle 0 \left| \prod_{i=1}^{n} \phi(x_i) \prod_{j=1}^{m} \phi^\dagger(y_j) \right| 0 \right\rangle \tag{7}$$
   vanish?
2. Compute the Wightman functions for $n + m \leq 6$.
3. Can you give a closed form expression for all n and m?
4. What changes for a neutral scalar field?