4. Problemset Relativistic Quantum Field Theory
November 15, 2017

Little Group / Second Quantization

4.1 Little Group Transformations

In the lecture, we have introduced the Lorentz transformations

$$\overline{\Lambda}^{\mu}{}_{\nu}(p) = \delta^{\mu}_{\nu} - \frac{(p+k)^{\mu}(p+k)_{\nu}}{kp+m^2} + 2\frac{p^{\mu}k_{\nu}}{m^2} \tag{1}$$

with $p^2 = k^2 = m^2$ that transform the reference momentum k to p

$$p^{\mu} = \overline{\Lambda}^{\mu}{}_{\nu}(p)k^{\nu}\,. \tag{2}$$

1. Determine the inverse transformation $(\overline{\Lambda}^{-1})^{\mu}{}_{\nu}(p)$ and verify by explicit computation that
$$(\overline{\Lambda}^{-1})^{\mu}{}_{\rho}(p)\overline{\Lambda}^{\rho}{}_{\nu}(p) = \delta^{\mu}_{\nu} \tag{3}$$

2. Other such transformations are
$$X^{\mu}{}_{\nu}(p) = \delta^{\mu}_{\nu} + \frac{(p-k)^{\mu}p_{\nu}}{pk} \tag{4a}$$
$$Y^{\mu}{}_{\nu}(p) = \delta^{\mu}_{\nu} + \frac{(p-k)^{\mu}k_{\nu}}{m^2} \tag{4b}$$
   (a) Show that $X(p)k = p$ and $Y(p)k = p$.
   (b) Describe the differences between $\overline{\Lambda}$, X and Y and discuss their relative merits.

4.2 Second Quantization

Consider the Fock space operators

$$B(M) = \sum_{\alpha,\beta} \int \widetilde{\mathrm{d}p}\, a^{\dagger}_{\alpha}(p)M_{\alpha\beta}a_{\beta}(p) \tag{5a}$$

$$F(M) = \sum_{\alpha,\beta} \int \widetilde{\mathrm{d}p}\, b^{\dagger}_{\alpha}(p)M_{\alpha\beta}b_{\beta}(p) \tag{5b}$$

derived from square matrices M using bosonic and fermionic creation and annihilation operators $a_{\alpha}(p)$, $a^{\dagger}_{\alpha}(p)$, $b_{\alpha}(p)$ and $b^{\dagger}_{\alpha}(p)$.

1. Compute the commutators

$$[B(M), B(N)]_- \tag{6a}$$
$$[F(M), B(N)]_- \tag{6b}$$

for two matrices M and N and interpret the result.

2. Assume that

$$MN = NM + zN \qquad \text{with } z \in \mathbf{C} \tag{7}$$

as matrix products and compute

$$\mathrm{e}^{B(M)} B(N) \mathrm{e}^{-B(M)} \,. \tag{8}$$

4.3 Matrix Elements

Consider the bosonic and fermionic two particle states

$$|f, g; +\rangle = \sum_{\alpha\beta} \int \widetilde{\mathrm{d}p}\widetilde{\mathrm{d}q} f^*_\alpha(p) g^*_\beta(q) a^\dagger_\alpha(p) a^\dagger_\beta(q) \,|0\rangle \tag{9a}$$
$$|f, g; -\rangle = \sum_{\alpha\beta} \int \widetilde{\mathrm{d}p}\widetilde{\mathrm{d}q} f^*_\alpha(p) g^*_\beta(q) b^\dagger_\alpha(p) b^\dagger_\beta(q) \,|0\rangle \,. \tag{9b}$$

1. Compute the inner products

$$\langle f, g; + | f', g'; + \rangle \tag{10a}$$
$$\langle f, g; - | f', g'; - \rangle \tag{10b}$$

2. Can you generalize you result to n-particle states?