3. Problemset Relativistic Quantum Field Theory
November 8, 2017

Poincaré Group & Algebra

3.1 Casimir Operators

In the lecture we have introduced the Pauli-Lubanski vector

$$W_\mu = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma} M^{\nu\rho} P^\sigma \,. \tag{1}$$

1. Use the Poincaré algebra to show that $W^2 = W_\mu W^\mu$ and $P^2 = P_\mu P^\nu$ commute with all generators of the Poincaré group

$$\left[P^\mu, P^2\right] = \left[P^\mu, W^2\right] = \left[M^{\mu\nu}, P^2\right] = \left[M^{\mu\nu}, W^2\right] = 0\,. \tag{2}$$

2. Show that the six independent operators

$$\left\{P_\mu P^\mu, P^1, P^2, P^3, W_\mu W^\mu, W^3\right\} \tag{3}$$

commute.

3.2 Scalars

Consider the representation of the Poincaré generators on scalar functions

$$P^\mu = \mathrm{i}\partial^\mu \tag{4a}$$
$$M^{\mu\nu} = -\mathrm{i}\left(x^\mu\partial^\nu - x^\nu\partial^\mu\right)\,. \tag{4b}$$

1. Compute the invariant length of the Pauli-Lubanski vector $W^2 = W_\mu W^\mu$ in this representation.

2. Interpret the result.

3.3 $\vec{L}$ and $\vec{K}$ revisited

Consider again the Lie algebra generated by the generators of rotations $\vec{L}$ and Lorentz boosts $\vec{K}$

$$[L_i, L_j] = \mathrm{i} \sum_{k=1}^{3} \epsilon_{ijk} L_k \tag{5a}$$

$$[L_i, K_j] = \mathrm{i} \sum_{k=1}^{3} \epsilon_{ijk} K_k \tag{5b}$$

$$[K_i, K_j] = -\mathrm{i} \sum_{k=1}^{3} \epsilon_{ijk} L_k \tag{5c}$$

and their linear combinations

$$\vec{A} = \frac{1}{2} \left(\vec{L} + \mathrm{i}\vec{K} \right) \tag{6a}$$

$$\vec{B} = \frac{1}{2} \left(\vec{L} - \mathrm{i}\vec{K} \right) \tag{6b}$$

1. Compute the commutation relations of $\vec{A}$ and $\vec{B}$.
2. Study the algebra generated by $\vec{A}$ and $\vec{B}$ using an analogy to angular momentum:
   (a) find two Casimir operators, and
   (b) find the irreducible representations labeled by two numbers.
3. From these, construct the irreducible representations of the algebra generated by $\vec{L}$ and $\vec{K}$.
4. Compute the action of $\vec{L}$ and $\vec{K}$ in a few representations with small eigenvalues of the Casimir operators explicitely.