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2. Problemset Relativistic Quantum Field Theory
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Rotations and Lorentz Transformations

2.1 SU(2) → SO(3) revisited

Consider the representation of vectors in $\mathbf{R}^3$ as traceless hermitian 2×2-matrices

$$\vec{x} \leftrightarrow \vec{x}\vec{\sigma} = \begin{pmatrix} x_3 & x_1 - \mathrm{i}x_2 \\ x_1 + \mathrm{i}x_2 & -x_3 \end{pmatrix} \tag{1}$$

1. Compute the transformed matrix X'

$$X \xrightarrow{\vec{\alpha}} X' = \mathrm{e}^{-\mathrm{i}\vec{\alpha}\vec{\sigma}/2} X \mathrm{e}^{\mathrm{i}\vec{\alpha}\vec{\sigma}/2} \tag{2}$$

   and the corresponding transformation

$$\vec{x} \xrightarrow{\vec{\alpha}} \vec{x'} = R\vec{x} \tag{3}$$

   with a real 3×3-matrix R as a function of $\vec{\alpha}$.

2. Show that $R \in \mathrm{SU}(3)$.

3. Compute the 3×3-matrix corresponding to a counter clockwise rotation around the axis α by the angle $|\alpha|$ and compare with R.

NB: You may start with the cases $\vec{\alpha} = (\alpha, 0, 0)$, $\vec{\alpha} = (0, \alpha, 0)$ to obtain a partial result (the case $\vec{\alpha} = (0, 0, \alpha)$ has been done in the lecture).

2.2 Lorentz Algebra

Define a antisymmetric 4×4-matrix of generators

$$M^{ij} = \sum_{k=1}^{3} \epsilon_{ijk} L_k \tag{4a}$$

$$M^{0i} = -M^{i0} = -K_i\,. \tag{4b}$$

in terms of six generators $\vec{L}$ and $\vec{K}$ with commutation relations

$$[L_i, L_j] = \mathrm{i} \sum_{k=1}^{3} \epsilon_{ijk} L_k \tag{5a}$$

$$[L_i, K_j] = \mathrm{i}\sum_{k=1}^{3}\epsilon_{ijk}K_k \tag{5b}$$

$$[K_i, K_j] = -\mathrm{i}\sum_{k=1}^{3}\epsilon_{ijk}L_k\,. \tag{5c}$$

1. Compute $[M^{\mu\nu}, M^{\rho\sigma}]$.

2.3 $\mathrm{SL}(2,\mathbf{C}) \to \mathcal{L}_+^\uparrow$

Consider the representation of vectors in $\mathbf{M} \cong \mathbf{R}^4$ as hermitian 2×2-matrices

$$x \mapsto X = x_\mu\sigma^\mu = x_0\mathbf{1} - \vec{x}\vec{\sigma} = \begin{pmatrix} x_0 - x_3 & -x_1 + \mathrm{i}x_2 \\ -x_1 - \mathrm{i}x_2 & x_0 + x_3 \end{pmatrix}\,. \tag{6}$$

1. Compute the transformed matrix X'

$$X \xrightarrow{\vec{\alpha}} X' = \mathrm{e}^{-\mathrm{i}\vec{\alpha}\vec{\sigma}/2}X\mathrm{e}^{\mathrm{i}\vec{\alpha}\vec{\sigma}/2} \tag{7}$$

   with *real* $\vec{\alpha} \in \mathbf{C}^3$. Show that $X \to X'$ corresponds to a rotation $\vec{x} \xrightarrow{\alpha} \vec{x'} = \Lambda\vec{x}$. You may (and should) use the result of the first problem.

2. Compute the transformed matrix X'

$$X \xrightarrow{\vec{\alpha}} X' = \mathrm{e}^{-\mathrm{i}\vec{\alpha}\vec{\sigma}/2}X\mathrm{e}^{\mathrm{i}\vec{\alpha}\vec{\sigma}/2} \tag{8}$$

   with *pure imaginary* $\vec{\alpha} = \mathrm{i}\vec{\beta} \in \mathbf{C}^3$ and $\vec{\beta} \in \mathbf{R}^3$. Show that $X \to X'$ corresponds to a Lorentz boost $\vec{x} \xrightarrow{\alpha} \vec{x'} = \Lambda\vec{x}$. *NB: You may start with the cases $\vec{\beta} = (\beta, 0, 0)$, $\vec{\beta} = (0, \beta, 0)$ and $\vec{\beta} = (0, 0, \beta)$ to obtain a partial result.*

3. What is the relative velocity of the two inertial frames?