1. Problemset Relativistic Quantum Field Theory
October 18, 2017

Groups, Algebras and Exponentials

1.1 Commutators

Consider the unitary matrix exponentials

$$R(\alpha) = \mathrm{e}^{\mathrm{i}\sum_a \alpha_a T_a} \tag{1}$$

with $\alpha \in \mathbf{R}^n$ and $\{T_a\}_{a=1,\ldots,n}$ a complete set of self adjoint $n \times n$ matrices.

1. Show by expansion to second order in the parameters α and β, that in order to be able to write

   $$R(\alpha)R(\beta) = R(\gamma) \tag{2}$$

   the matrices $\{T_a\}_{a=1,\ldots,n}$ must close under commutation

   $$[T_a, T_b] = \mathrm{i}\sum_c f_{abc}T_c \tag{3}$$

   even if they don't close under multiplication

   $$T_aT_b \neq \mathrm{i}\sum_c f'_{abc}T_c\,. \tag{4}$$

   Hint: compute

   $$\mathrm{i}\sum_a \gamma_a T_a = \ln\left(R(\alpha)R(\beta)\right) \tag{5}$$

   to second order in the parameters α and β.

2. Continue the expansion to third order in the parameters α and β. Do nontrivial new conditions appear?

1.2 $\mathrm{SL}(2, \mathbf{C})$

Use the Pauli matrices to parametrize a general complex 2×2-matrix M by four complex numbers $(a_0, \vec{a})$

$$M(a_0, \vec{a}) = a_0 \mathbf{1} + \vec{a}\vec{\sigma} = a_0 \mathbf{1} + a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3 = \begin{pmatrix} a_0 + a_3 & a_1 - \mathrm{i}a_2 \\ a_1 + \mathrm{i}a_2 & a_0 - a_3 \end{pmatrix} \tag{6}$$

1. Show that
$$\exp\left(\mathrm{i}M(a_0, \vec{a})\right) = \mathrm{e}^{\mathrm{i}a_0} M\left(\cos a, \mathrm{i}\frac{\sin a}{a}\,\vec{a}\right) \qquad \text{with} \quad a = \sqrt{\vec{a}^{\,2}}\,. \tag{7}$$

2. Can all $M(a_0, \vec{a})$ be written as $M(b_0, \vec{b})$ with four suitable complex numbers $(b_0, \vec{b})$? If not, which subset can?

3. What is the condition on $(a_0, \vec{a})$ for $M(a_0, \vec{a})$ to be unimodular, i. e. for $\det M(a_0, \vec{a}) = 1$?

4. Does the set of all $M(a_0, \vec{a})$ form a group? If not, can you find subsets that do?

5. What is the condition on $(a_0, \vec{a})$ for $\exp\left(\mathrm{i}M(a_0, \vec{a})\right)$ to be unimodular, i. .e. for $\exp\left(\mathrm{i}M(a_0, \vec{a})\right) = 1$?

6. Does the set of all $\exp\left(\mathrm{i}M(a_0, \vec{a})\right)$ form a group? If not, can you find subsets that do?

1.3 Hausdorff's Formula

Prove Hausdorff's formula

$$\mathrm{e}^{A} B \mathrm{e}^{-A} = \mathrm{e}^{\mathrm{ad}_A} B \tag{8}$$

with the adjunction operator ad_A

$$\mathrm{ad}_A B = [A, B] \tag{9}$$

for arbitrary operators and matrices.

Hint: replace $A \to tA$ (with $t \in \mathbf{R}$) on both sides and show that both sides of the equation solve the same initial value problem.