

Algebra and Dynamics of Quantum Systems

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Abstract

- General mathematical description of a physical system: C^* -algebras, spectra and states
- Mathematical description of quantum systems: states and representations and the GNS construction.
- Single particle quantum mechanics: Weyl algebra, Stone-von Neumann theorem, Schrödinger wave functions, minimum uncertainty states.
- Quantum dynamics: Schrödinger equation, Hamiltonian, self-adjointness, examples.
- Mathematical description of infinitely extended systems: Haag's theorem.
- Symmetries in quantum mechanics: Wigner's theorem
- Symmetry breaking: Goldstone's theorem, Higgs mechanism, superconductivity.

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Vorbemerkung

Dieses Manuskript ist mein persönliches Vorlesungsmanuskript, an vielen Stellen nicht ausformuliert und kann jede Menge Fehler enthalten. Es handelt sich hoffentlich um weniger Denk- als Tippfehler, trotzdem kann ich deshalb ich keine Verantwortung für Fehler übernehmen. Zeittranslationsinvarianz ist natürlich auch nicht gegeben ...

Dennoch, oder gerade deshalb, bin ich für alle Korrekturen und Vorschläge dankbar!

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Aktuelle Informationen

Vorlesungs-URL zu lang, einfach auf Links auf <http://physik.uni-wuerzburg.de/ohl/> folgen.

— 1 —

INTRODUCTION

1.1 Literature

Lecture 01: Wed, 17.10.2018

1.1.1 Close to the lecture

- Franco Strocchi, *An Introduction to the Mathematical Structure of Quantum Mechanics*, World Scientific, 2nd ed., 2008.
- Franco Strocchi, *Symmetry Breaking*, Springer, 2nd ed., 2007. **Available online:** <http://www.springerlink.com/content/978-3-540-73592-2/>

1.1.2 Quantum Field Theory

- Rudolf Haag, *Local Quantum Physics: Fields, Particles, Algebras*, Springer, 2nd ed., 1996.

1.1.3 C^* -Algebras

- Ola Bratteli, Derek Robinson, *Operator Algebras and Quantum Statistical Mechanics, Vols. I, II*, Springer, 2nd ed., 2002 (these used to be available free of charge as PDFs on Ola Bratteli's homepage <http://folk.uio.no/bratteli/>, but the page has disappeared).
- Jacques Dixmier, *C^* -Algebras*, North-Holland Mathematical Library, Vol. 15, 1977.
- Masamichi Takesaki, *Theory of Operator Algebra, Vol. 1*, Springer, 2001

1.1.4 Functional Analysis

- Michael Reed, Barry Simon, *Methods of modern mathematical physics, Volume 1: Functional analysis*, Academic Press, 2nd ed., 1981.

1.1.5 Historical

- John von Neumann, *Mathematical Foundations of Quantum Mechanics*, Princeton University Press 1932.

1.2 Naive Formulation of Quantum Mechanics

Schrödinger wave functions are usually the first formulation introduced in Quantum Mechanics (QM) lectures. Its ingredients are

- The state of a physical system is described by a wave function, which is assumed to be square integrable and normalized. For a single particle it is a function of the space points and time

$$\begin{aligned} \psi : \mathbf{R}^3 \times \mathbf{R} &\rightarrow \mathbf{C} \\ (\vec{x}, t) &\mapsto \psi(\vec{x}, t) \end{aligned} \quad (1.1)$$

and the probability of finding the particle in a domain $D \in \mathbf{R}^3$ is given by

$$p_D(t) = \int_D d^3x |\psi(\vec{x}, t)|^2. \quad (1.2)$$

where ψ has been normalized $\int_{\mathbf{R}^3} d^3x |\psi(\vec{x}, t)|^2 = 1$. In other words, $|\psi(\vec{x}, t)|^2$ is interpreted as a probability density.

- Superposition principle: if $\psi_{1,2}$ are states, $\psi = c_1\psi_1 + c_2\psi_2$ with $|c_1|^2 + |c_2|^2 = 1$ is also a state.
- Other observables are constructed by a correspondence principle from a classical dynamical system

$$\text{position: } \vec{x}(t) \Leftrightarrow \hat{x} \quad (1.3)$$

$$\text{momentum: } \vec{p}(t) \Leftrightarrow \hat{p} \quad (1.4)$$

where

$$(\hat{x}\psi)(x, t) = x\psi(x, t) \quad (1.5)$$

$$(\hat{p}\psi)(x, t) = -i\vec{\nabla}\psi(x, t). \quad (1.6)$$

Note:

- The choice of \hat{x} is intuitively clear, because it yields the correct expectation values.
- The choice of \hat{p} can be motivated by each of
 - * the de Broglie relation $\vec{p} = \hbar\vec{k}$ for plain waves
 - * the resulting Canonical Commutation Relations (CCR) $[\hat{x}_i, \hat{p}_j] = i\delta_{ij}$ reproduce the Heisenberg uncertainty relation $\Delta x \Delta p \geq 1/2$
 - * as the generator of translations (cf. [Wei13, Ohl18])
 or any combination of these arguments.

- the dynamics of the system is described by the Schrödinger equation

$$i\frac{\partial}{\partial t}\psi(\vec{x}, t) = H\psi(\vec{x}, t) \quad (1.7)$$

with a Hamiltonian H that is motivated by the correspondence principle

$$H = \frac{1}{2m}p^2 + V(\vec{x}) = -\frac{1}{2m}\Delta + V(\vec{x}). \quad (1.8)$$

- physical interpretation:
 - possible results of measuring an observable O : the spectrum of the corresponding operator
 - expectation value for multiple measurements of an observable O :

$$\langle O \rangle = \int d^3x \bar{\psi}(\vec{x}, t) O \psi(\vec{x}, t) \quad (1.9)$$

and the probability of finding the particle in a domain D can be interpreted as the expectation value for the characteristic function χ_D of D

$$p(D) = \int_D d^3x |\psi(\vec{x}, t)|^2 = \int d^3x \bar{\psi}(\vec{x}, t) \chi_D(\vec{x}) \psi(\vec{x}, t) = \langle \chi_D \rangle, \quad (1.10)$$

where ψ has been normalized $\int_{\mathbf{R}^3} d^3x |\psi(\vec{x}, t)|^2 = 1$.

1.3 Problems

The spectral theorem is required to make the interpretation consistent and to define functions of operators like

$$f(A) = \int da f(a) |a\rangle\langle a|, \quad (1.11)$$

but

1. The **CCR** $[x, p] = i$ can *not* be realized with bounded operators (\rightarrow *exercise*).
2. *Hellinger-Toeplitz theorem*: an everywhere defined symmetric operator, i. e.

$$\forall \psi, \phi \in \mathcal{H} : (\psi, A\phi) = (A\psi, \phi) \quad (1.12)$$

on a Hilbert \mathcal{H} space is bounded (see, e. g. [RS80], p. 84, corollary to the closed graph theorem III.12).

Therefore, we have to be careful about the domain of our operators:

1. there are square integrable $\psi : \mathbf{R} \rightarrow \mathbf{C}$ for which $(x\psi) : \mathbf{R} \rightarrow \mathbf{C}$ is not square integrable
2. the operators x , p and $[x, p]$ all have different domains.

In fact, the development of a mathematically rigorous spectral theory for unbounded operators was, to a large extent, driven by these technical difficulties of quantum mechanics.

Instead of retracing these historical steps starting from Schrödinger's and Heisenberg's heuristics, we shall go back and ask ourselves what is the *most general* framework for a mathematical description of physical systems.

1.4 Mathematical Description of Physical Systems

We will start with the intuitive example of classical mechanics and later relax some assumptions to obtain a framework that encompasses quantum mechanics.

1.4.1 Classical Mechanics

Canonical Phase Space

kinematics: the kinematical structure is determined by the *state* of the system and the *observables* that can be measured:

state: in an idealized classical mechanical system with n degrees of freedom (**d.o.f.**), the state is given by a point x in phase space Γ , i. e. in a chart

$$x = (q, p) = (q_1, \dots, q_n, p_1, \dots, p_n) \in \Gamma \quad (1.13)$$

where Γ is often a cotangent bundle T^*Q over the configuration space $Q \ni q$.

observables: we can measure all polynomials of the phase space variables q and p and this should extend to the closure of the polynomials in the $\|\cdot\| = \sup_{\Gamma} |\cdot|$ -norm, i. e. the continuous functions on phase space: $C(\Gamma)$. Sometimes it is useful to allow complex coefficients for the polynomials and complex valued continuous functions.

dynamics: the time evolution of the state $x = (q, p)$ of a classical mechanical system is described by a flow Φ on phase space

$$\begin{aligned} \Phi : \mathbf{R} \times \Gamma &\rightarrow \Gamma \\ (t, q, p) &\mapsto \Phi_t(q, p) = (q(t), p(t)) \end{aligned} \quad (1.14)$$

(see section 5.4.2 on page 156 of [Ohl16] for further discussions). This induces the time evolution of the observables by

$$\begin{aligned} \Phi^* : \mathbf{R} \times C(\Gamma) &\rightarrow C(\Gamma) \\ (t, f) &\mapsto \Phi_t^*(f) = f \circ \Phi_t \end{aligned} \quad (1.15)$$

i. e.

$$\Phi_t^*(f)(q, p) = f(q(t), p(t)). \quad (1.16)$$

This flow is a solution of the canonical equations of motion

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \quad (1.17a)$$

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \quad (1.17b)$$

with a suitable Hamiltonian $H : \Gamma \rightarrow \mathbf{R}$. Using the antisymmetric Poisson brackets

$$\{q_i, p_j\} = \delta_{ij} \quad (1.18a)$$

$$\{q_i, q_j\} = \{p_i, p_j\} = 0 \quad (1.18b)$$

we may also write more uniformly

$$\frac{d}{dt}q_i = \{q_i, H\} \quad (1.19a)$$

$$\frac{d}{dt}p_i = \{p_i, H\} \quad (1.19b)$$

and find with the chain rule

$$\frac{d}{dt}\Phi_t^*(f) = \{\Phi_t^*(f), H\} \quad (1.20)$$

for all observables $f : \Gamma \rightarrow \mathbf{R}$.

Algebras of Observables

The space $C(\Gamma)$ of all observables carries a natural *abelian *-algebra* structure

$$\forall \alpha, \beta \in \mathbf{C} : (\alpha f + \beta g)(x) = \alpha f(x) + \beta g(x) \quad (1.21a)$$

$$(fg)(x) = f(x)g(x) = (gf)(x) \quad (1.21b)$$

$$(f^*)(x) = \overline{f(x)} \quad (1.21c)$$

with $f : x \mapsto 1$ as identity. In general we have

Definition 1.1 (algebra). An *algebra* \mathcal{A} over a field \mathbf{K} is a vector space over \mathbf{K} together with a bilinear internal binary operation

$$\begin{aligned} \cdot : \mathcal{A} \times \mathcal{A} &\rightarrow \mathcal{A} \\ (x, y) &\mapsto xy, \end{aligned} \quad (1.22)$$

i. e. $\forall x, y, z \in \mathcal{A}, \alpha, \beta \in \mathbf{K}$

$$z(\alpha x + \beta y) = \alpha(zx) + \beta(zy) \quad (1.23a)$$

$$(\alpha x + \beta y)z = \alpha(xz) + \beta(yz) \quad (1.23b)$$

$$(\alpha x)(\beta y) = (\alpha\beta)(xy). \quad (1.23c)$$

The algebra \mathcal{A} is called

- *associative*, if and only if (iff) $\forall x, y, z \in \mathcal{A} : (xy)z = x(yz)$,
- *commutative* or *abelian*, iff $\forall x, y \in \mathcal{A} : xy = yx$ and
- *unital* or with identity, iff $\exists_1 e \in \mathcal{A} : \forall x \in \mathcal{A} : ex = xe = x$.

Note that some authors reserve the term algebra for associative algebras.

In the following, we will only use algebras over \mathbf{C} . Most of the algebras will be associative and many unital. The more interesting cases will not be commutative, however.

Definition 1.2 (**-algebra / involutive algebra*). A **-algebra* or *involutive algebra* $(\mathcal{A}, *)$ is an associative algebra \mathcal{A} over \mathbf{C} with an *antiautomorphism* or *involution*

$$\begin{aligned} * : \mathcal{A} &\rightarrow \mathcal{A} \\ x &\mapsto x^* \end{aligned} \tag{1.24}$$

i. e. $\forall x, y \in \mathcal{A}, \alpha \in \mathbf{C}$

$$(x^*)^* = x \tag{1.25a}$$

$$(x + y)^* = x^* + y^* \tag{1.25b}$$

$$(\alpha x)^* = \bar{\alpha} x^* \tag{1.25c}$$

$$(xy)^* = y^* x^* \tag{1.25d}$$

where $\bar{\alpha}$ is the usual complex conjugate of α in \mathbf{C} .

Note that it is in principle possible to use associative algebras over fields other than \mathbf{C} , but this is outside of our scope.

Definition 1.3 (normal, self-adjoint, positive, unitary). An element x of a *-algebra is called

- *normal*, iff $x^*x = xx^*$,
- *self-adjoint*, iff $x^* = x$,
- *positive*, iff $\exists y \in \mathcal{A} : x = y^*y$,
- an *isometry*, iff $x^*x = e$,
- *unitary*, iff $x^*x = xx^* = e$.

The last two only apply to unital algebras, of course.

Lemma 1.4. *A positive element x of a $*$ -algebra is self-adjoint. Furthermore, self-adjoint and unitary elements are normal.*

Proof. The first statement follows from $x^* = (y^*y)^* = y^*(y^*)^* = y^*y = x$ and the second and third statements are trivial. \square

In $C(\Gamma)$, there is a natural norm

$$\|f\| = \sup_{x \in \Gamma} |f(x)| \quad (1.26)$$

and $C(\Gamma)$ is by construction complete w.r.t. to this norm, i.e. a Banach space. Since

$$\forall f, g \in C(\Gamma) : \|fg\| \leq \|f\| \|g\| \quad (1.27)$$

the product is continuous w.r.t. to the norm topology in both factors and thus we have a *Banach $*$ -algebra*.

Definition 1.5 (normed algebra). A *normed algebra* $(\mathcal{A}, \|\cdot\|)$ is an associative algebra \mathcal{A} together with a norm $\|\cdot\|$ that satisfies

$$\forall x, y \in \mathcal{A} : \|xy\| \leq \|x\| \|y\|. \quad (1.28)$$

Therefore the multiplication is continuous w.r.t. to the norm topology in both factors.

Definition 1.6 (Banach algebra). A *Banach algebra* is a normed algebra $(\mathcal{A}, \|\cdot\|)$ that is complete w.r.t. to the norm $\|\cdot\|$.

Definition 1.7 (normed $*$ -algebra). A *normed $*$ -algebra* $(\mathcal{A}, *, \|\cdot\|)$ is a $*$ -algebra $(\mathcal{A}, *)$ together with a norm $\|\cdot\|$ that satisfies

$$\forall x \in \mathcal{A} : \|x^*\| = \|x\|. \quad (1.29)$$

Definition 1.8 (Banach $*$ -algebra). A *Banach $*$ -algebra* is a normed $*$ -algebra $(\mathcal{A}, *, \|\cdot\|)$ that is complete w.r.t. to the norm $\|\cdot\|$.

Finally we find that

$$\forall f \in C(\Gamma) : \|f^*f\| = \|f^*\| \|f\| \quad (1.30)$$

and $C(\Gamma)$ turns out to be a unital C^* -algebra:

Definition 1.9 (C^* -algebra). A C^* -algebra is a Banach $*$ -algebra $(\mathcal{A}, *, \|\cdot\|)$ that satisfies the C^* -condition

$$\forall x \in \mathcal{A} : \|x^*x\| = \|x\|^2. \quad (1.31)$$

Remark 1.10. The C^* -condition (1.31) implies (1.29), i. e. $\forall x \in \mathcal{A} : \|x^*\| = \|x\|$.

Proof. On one hand

$$\|x\|^2 \stackrel{(1.31)}{=} \|x^*x\| \leq \|x^*\| \|x\| \quad (1.32)$$

i. e.

$$\|x\| \leq \|x^*\| \quad (1.33)$$

and on the other

$$\|x^*\|^2 \stackrel{(1.31)}{=} \|(x^*)^*x^*\| = \|xx^*\| \leq \|x\| \|x^*\| \quad (1.34)$$

i. e.

$$\|x^*\| \leq \|x\|. \quad (1.35)$$

□

Corollary 1.11. The C^* -condition (1.31) is equivalent to $\|x^*x\| = \|x\| \|x^*\|$.

We will later see that the C^* -condition (1.31) is very strong and has important consequences. Therefore we should look at the example motivating the definition.

Theorem 1.12 (C^* -algebra of bounded operators). The set $L(\mathcal{H})$ of bounded operators on a Hilbert space \mathcal{H} is a C^* -algebra

Proof. $L(\mathcal{H})$ is naturally a linear space and we can define a norm via

$$\|A\| = \sup_{\psi \in \mathcal{H}, \|\psi\|=1} \{\|A\psi\|\}. \quad (1.36)$$

For every $A \in L(\mathcal{H})$, we can define the adjoint operator A^* , thus turning $L(\mathcal{H})$ into a Banach $*$ -algebra. Finally, using the Cauchy-Schwarz Inequality (CSI)

$$|(\psi, \phi)| \leq \|\psi\| \|\phi\|, \quad (1.37)$$

we can show that the C^* -condition (1.31) holds for $L(\mathcal{H})$:

$$\begin{aligned} \|A\|^2 &= \sup_{\psi \in \mathcal{H}, \|\psi\|=1} \{(A\psi, A\psi)\} = \sup_{\psi \in \mathcal{H}, \|\psi\|=1} \{(\psi, A^*A\psi)\} \\ &\stackrel{\text{CSI}}{\leq} \sup_{\psi \in \mathcal{H}, \|\psi\|=1} \{\|A^*A\psi\|\} = \|A^*A\| \leq \|A^*\| \|A\| = \|A\|^2. \end{aligned} \quad (1.38)$$

□

In fact, the converse is also true

Theorem 1.13. *Every C^* -algebra is isomorphic to a norm-closed self-adjoint algebra¹ of bounded operators on a suitable Hilbert space.*

The proof of this theorem will not be given at this point (see section 2.3.4 of [BR02]).

A very important special case is the following characterization of *abelian* C^* -algebras. It shows that our example $C(\Gamma)$ is already the general case:

Theorem 1.14. *Every abelian C^* -algebra is isomorphic to the algebra $C_0(X)$ of continuous functions on a locally compact Hausdorff space X , vanishing at infinity.*

This theorem has the profound consequence that geometry and commutative algebra are two sides of the same coin. The maximal ideals of an algebra of functions on a space X are the functions vanishing at a point $x \in X$. Therefore the points of a space can be identified with the maximal ideals of an abelian C^* -algebra. Indeed, some aspects of the study of non-abelian C^* -algebras are called “noncommutative geometry”. Again, the proof of this theorem will not be given at this point (see section 2.3.5 of [BR02]).

States as Linear Functionals

In the real world, it is *impossible* to determine the state of a system as a single point x in phase space. One rather makes measurements with finite errors and repeated measurements may or may not be correlated, depending on the experimental situation.

Given a set

$$\{\mu_n(f, \omega)\}_{n=1, \dots, N} \quad (1.39)$$

of N measurements of an observable f of a system in the state ω , we can define an estimator

$$\langle f \rangle_{\omega, N} = \frac{1}{N} \sum_{n=1}^N \mu_n(f, \omega) \quad (1.40)$$

for the *expectation*

$$\omega(f) = \lim_{N \rightarrow \infty} \langle f \rangle_{\omega, N} \quad (1.41)$$

of f in the state ω . We can also determine higher moments, e. g.

$$(\Delta_\omega f)^2 = \omega((f - \omega(f))^2), \quad (1.42)$$

¹Note that “self-adjoint algebra” does *not* mean a algebra of self-adjoint operators, but a algebra which includes the adjoint of every element.

to measure experimental uncertainties, but we can *in principle* never do better.

From the definition of the expectation $\omega(f)$, we immediately see that they are linear

$$\forall \alpha, \beta \in \mathbf{C}, f, g \in \mathcal{A} : \omega(\alpha f + \beta g) = \alpha \omega(f) + \beta \omega(g) \quad (1.43a)$$

and (ignoring systematic errors) positive

$$\forall f \in \mathcal{A} : \omega(f^* f) \geq 0. \quad (1.43b)$$

Using linearity and positivity

$$\begin{aligned} \forall \alpha, \beta \in \mathbf{C} : 0 &\leq \omega((\alpha f + \beta g)^*(\alpha f + \beta g)) \\ &= |\alpha|^2 \omega(f^* f) + \bar{\beta} \alpha \omega(g^* f) + \bar{\alpha} \beta \omega(f^* g) + |\beta|^2 \omega(g^* g) \\ &= (\bar{\alpha} \quad \bar{\beta}) \begin{pmatrix} \omega(f^* f) & \omega(f^* g) \\ \omega(g^* f) & \omega(g^* g) \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \end{aligned} \quad (1.44)$$

we see that the matrix

$$M = \begin{pmatrix} \omega(f^* f) & \omega(f^* g) \\ \omega(g^* f) & \omega(g^* g) \end{pmatrix} \quad (1.45)$$

must be positive, i. e. have only real and non-negative eigenvalues

$$\lambda_{1,2} = \frac{1}{2} \operatorname{tr} M \pm \frac{1}{2} \sqrt{\operatorname{tr}^2 M - 4 \det M}. \quad (1.46)$$

Therefore

$$0 \leq \det M = \omega(f^* f) \omega(g^* g) - \omega(g^* f) \omega(f^* g) \in \mathbf{R} \quad (1.47)$$

and we have established the *CSI*

$$\omega(g^* f) = \overline{\omega(f^* g)} \quad (1.48a)$$

$$|\omega(f^* g)| \leq \sqrt{\omega(f^* f)} \sqrt{\omega(g^* g)}. \quad (1.48b)$$

From the *CSI*

$$|\omega(f)| \leq \sqrt{\omega(\mathbf{1})} \sqrt{\omega(f^* f)}. \quad (1.49)$$

we infer that

$$\exists f \in \mathcal{A} : \omega(f) \neq 0 \implies \sqrt{\omega(\mathbf{1})} > 0 \quad (1.50)$$

Therefore all nontrivial states can be normalized

$$\omega_{\text{norm}}(f) = \frac{\omega(f)}{\omega(\mathbf{1})} \quad (1.51)$$

with $\omega_{\text{norm}}(\mathbf{1}) = 1$. Below, in section 4, when we have developed more tools, we will be able to prove that every positive linear functional is continuous. This is an example, where a physical requirement (positivity) induces a functional analytic property (continuity).

Just as the flow $\Phi : \mathbf{R} \times \Gamma \rightarrow \Gamma$ on phase space induces a flow $\Phi^* : \mathbf{R} \times C(\Gamma) \rightarrow C(\Gamma)$ on the space of observables, we can also induce a flow on the states in Ω

$$\begin{aligned} \Phi_* : \mathbf{R} \times \Omega &\rightarrow \Omega \\ (t, \omega) &\mapsto \omega_t \end{aligned} \quad (1.52a)$$

with

$$\omega_t(f) = \omega(\Phi_t^*(f)) = \omega(f \circ \Phi_t). \quad (1.52b)$$

Note that the idealized unphysical case of an exactly determined point in phase space can be included in this framework by considering states $\omega_{q,p}$ that correspond to *Dirac measures*

$$\begin{aligned} \omega_{q,p} : C(\Gamma) &\rightarrow \mathbf{C} \\ f &\mapsto f(q, p). \end{aligned} \quad (1.53)$$

For these,

$$\begin{aligned} (\Delta_{\omega_{q,p}} f)^2 &= \omega_{q,p}((f - \omega_{q,p}(f))^2) \\ &= \omega_{q,p}((f - f(q, p))^2) = ((f - f(q, p))^2)(q, p) = 0. \end{aligned} \quad (1.54)$$

Algebraic Dynamics

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Thus we have abstracted the kinematical structure as

- the observables A of a classical system form an abelian C^* -algebra \mathcal{A} and
- the states ω of a classical system are the *normalized positive linear functionals* $\omega : \mathcal{A} \rightarrow \mathbf{C}$, i. e. a subset of the dual space \mathcal{A}^* .

On this level of abstraction, the flows Φ are replaced by an abelian one-parameter group $\{\alpha_t\}_{t \in \mathbf{R}}$ of C^* -automorphisms

$$\begin{aligned} \alpha : \mathbf{R} \times \mathcal{A} &\rightarrow \mathcal{A} \\ (t, x) &\mapsto \alpha_t(x) \end{aligned} \quad (1.55a)$$

with

$$\alpha_t(\alpha_{t'}(x)) = (\alpha_t \circ \alpha_{t'})(x) = \alpha_{t+t'}(x) = \alpha_{t'}(\alpha_t(x)). \quad (1.55b)$$

Definition 1.15 (C^* -homomorphism). A C^* -homomorphism $h : \mathcal{A} \rightarrow \mathcal{B}$ is a map between two C^* -algebras \mathcal{A} and \mathcal{B} that preserves the C^* -algebra structure. It is of course linear and $\forall x, y \in \mathcal{A}$

$$(h(x))^* = h(x^*) \quad (1.56a)$$

$$h(xy) = h(x)h(y). \quad (1.56b)$$

This in turn induces an abelian one-parameter group of transformations on the states:

$$\begin{aligned} \alpha^* : \mathbf{R} \times \Omega &\rightarrow \Omega \\ (t, \omega) &\mapsto \alpha_t^*(\omega) \end{aligned} \quad (1.57a)$$

with

$$\alpha_t^*(\omega)(A) = \omega(\alpha_t(A)). \quad (1.57b)$$

1.4.2 General Physical Systems

Using as inspiration the algebraic structure we have just uncovered behind the description of classical systems, we will now generalize it *without* assuming the smooth phase space structure characteristic for classical systems.

Observables

For All Practical Purposes (**FAPP**), a physical system is nothing but a set \mathcal{O}_0 of *observables* $A \in \mathcal{O}_0$, which can be measured by appropriate experimental devices. This set \mathcal{O}_0 is not unstructured, there will relations among the observables.

Given any device measuring $A \in \mathcal{O}_0$ and any $\lambda \in \mathbf{R}$, one can construct a device measuring λA by rescaling the device. One does not even worry about finite resources, because the actual device need not be physically rescaled, just the measured value. Therefore we find

$$\forall A \in \mathcal{O}_0, \lambda \in \mathbf{R} : \lambda A \in \mathcal{O}_0. \quad (1.58)$$

One can define the square A^2 of an observable A by squaring the measured value. This extends to positive powers A^n and A^0 can consistently be defined as the observable $\mathbf{1}$ that always has the value 1. Also, linear combinations $\lambda A^n + \mu A^m$ can be defined and therefore arbitrary polynomials in one observable. Note that we can not yet define functions of observables by infinite series or other limits of sequences of observables, because we have no topology and no notion of convergence.

If an observable $A \in \mathcal{O}_0$ only takes positive values, we can find another observable $B \in \mathcal{O}_0$ with $A = B^2$, by taking the square root of the measured values.

Note that we do *not* assume that we can measure multivariate polynomials in different observables, since this would require the *simultaneous* measurement of observables, which would lead us back to classical physics immediately.

States

Different *states* ω of a physical system correspond to different measured values for the observables. However, we can not say that an observable “has” a particular value, we can only estimate an *expectation* for the result of a measurement by averaging the values of repeated measurements. The expectation value for the observable $A \in \mathcal{O}_0$ when the system is in the state ω will be called $\omega(A)$.

Assuming that the set \mathcal{O}_0 of the observables of the system is complete, a state ω is completely determined by the expectations of all observables

$$\{\omega(A)\}_{A \in \mathcal{O}_0}, \quad (1.59)$$

i. e. ω is a well defined map from \mathcal{O}_0 to the real numbers

$$\omega : \mathcal{O}_0 \rightarrow \mathbf{R}. \quad (1.60)$$

Since ω is defined as an average of measurements, it is clear that the expectation of a scaled observable is the scaled expectation

$$\forall A \in \mathcal{O}_0, \lambda \in \mathbf{R} : \omega(\lambda A) = \lambda \omega(A) \quad (1.61)$$

and we also have linearity in powers²

$$\forall A \in \mathcal{O}_0, \lambda, \mu \in \mathbf{R}, n, m \in \mathbf{N} : \omega(\lambda A^n + \mu A^m) = \lambda \omega(A^n) + \mu \omega(A^m). \quad (1.62)$$

Above, we have said that a state is uniquely determined by measuring $\omega(A)$ for all observables $A \in \mathcal{O}_0$.

The converse is also true: if two observables have the same expectation in *all* the states, they are indistinguishable, **FAPP**. This creates an equivalence relation among observables

$$A \sim B \Leftrightarrow \forall \omega \in \Omega : \omega(A) = \omega(B) \quad (1.63)$$

²NB: In the lecture of Oct. 24, 2018, I erroneously wrote $\omega(A^n) = (\omega(A))^n$. This is *not* correct: we have defined the n -th power of A for each individual measurement μ as $\mu(A^n, \omega) = (\mu(A, \omega))^n$. This does *not* imply the analogous formula for the expectation values $\omega(A^n)$!

and we will in the following replace \mathcal{O}_0 by the equivalence classes w. r. t. this relation $\mathcal{O}_1 = \mathcal{O}_0 / \sim$. This has the immediate consequence that there is at most one observable $\mathbf{1} \in \mathcal{O}_1$ with

$$\forall \omega \in \Omega : \omega(\mathbf{1}) = 1. \quad (1.64)$$

And we see that all ω are *normalized* functionals on \mathcal{O}_1 . Furthermore, we can identify

$$\forall A \in \mathcal{O}_1 : A^0 = \mathbf{1} \quad (1.65)$$

and we necessarily have

$$\forall A \in \mathcal{O}_1 : \mathbf{1}A = A^0A = A^0A^1 = A^1 = A = A\mathbf{1}, \quad (1.66)$$

i. e. $\mathbf{1}$ is the unique identity element in \mathcal{O}_1 .

For all positive observables $A = B^2 \in \mathcal{O}_1$, the expectation of its values must also be positive, i. e.

$$\forall \omega \in \Omega : \omega(A) = \omega(B^2) \geq 0 \quad (1.67)$$

and since the states characterize the observables completely

$$\forall \omega \in \Omega : \omega(A) \geq 0 \Leftrightarrow A \geq 0. \quad (1.68)$$

C-Algebra Structure*

For any physically realizable apparatus, the possible measurement values are bounded. Since ω is normalized, this allows to define a natural norm for any observable $A \in \mathcal{O}_1$

$$\|A\| = \sup_{\omega \in \Omega} |\omega(A)| < \infty. \quad (1.69)$$

From this definition and (1.61), we have

$$\|\lambda A\| = |\lambda| \|A\|. \quad (1.70)$$

Since the states characterize the observables completely, we also have

$$\|A\| = 0 \Leftrightarrow A = 0. \quad (1.71)$$

The first non-obvious property is

Lemma 1.16.

$$\forall A \in \mathcal{O}_1 : \|A^2\| = \|A\|^2. \quad (1.72)$$

Proof. By definition

$$\forall \omega \in \Omega : \omega(\|A\| \mathbf{1} \pm A) = \|A\| \pm \omega(A) \geq 0 \quad (1.73)$$

i. e.

$$\|A\| \mathbf{1} \pm A \geq 0. \quad (1.74)$$

Therefore

$$\|A\|^2 \mathbf{1} - A^2 = (\|A\| \mathbf{1} + A)(\|A\| \mathbf{1} - A) \geq 0, \quad (1.75)$$

i. e.

$$\forall \omega \in \Omega : \|A\|^2 - \omega(A^2) \geq 0 \quad (1.76)$$

and

$$\|A\|^2 \geq \|A^2\|. \quad (1.77)$$

Also, from

$$0 \leq (\|A\| \mathbf{1} \pm A)^2 = \|A\|^2 \mathbf{1} + A^2 \pm 2\|A\|A \quad (1.78)$$

we have

$$\forall \omega \in \Omega : 2\|A\|\omega(A) \leq \|A\|^2 + \omega(A^2) \leq \|A\|^2 + \|A^2\|, \quad (1.79)$$

and

$$2\|A\|\|A\| \leq \|A\|^2 + \|A^2\|, \quad (1.80)$$

i. e.

$$\|A\|^2 \leq \|A^2\|. \quad (1.81)$$

□

Since the states characterize the observables completely, we can use them to attempt to define sums of observables

$$\forall A, B \in \mathcal{O}_1 : \exists_1 A + B : \omega(A + B) = \omega(A) + \omega(B),$$

but $A + B$ might *not* be an element of the original \mathcal{O}_1 . If this is the case, however, we will extend \mathcal{O}_1 to include the newly defined $A + B$, its powers and their sums

$$\forall A, B \in \mathcal{O}_+ : \exists_1 A + B \in \mathcal{O}_+ : \forall \omega \in \Omega : \omega(A + B) = \omega(A) + \omega(B). \quad (1.82)$$

Note that it is *not* necessary to measure A and B simultaneously for (1.82) to be well defined. The estimators for the expectationa $\omega(A)$ are obtained from repeated measurements of identical copies of the system prepared in the state ω . This can be done for different observables as well and the sum

defined from the sum of the measurements. This is *not* possible for products or other non-linear combinations.

So far, we have constructed from a set of observables \mathcal{O}_0 a vector space \mathcal{O}_+ with a linear structure defined by the set of all states ω . By the definition of the norm (1.69) on the original \mathcal{O}_0 , we can directly extend it to the sums, satisfying the triangle inequality

$$\forall A, B \in \mathcal{O}_+ : \|A + B\| \leq \|A\| + \|B\|. \quad (1.83)$$

Therefore \mathcal{O}_+ is a normed vector space and by completing it in the norm $\|\cdot\|$, we can turn it into a Banach space \mathcal{O} . However, while the states are, by construction (1.82), well defined on the sums of observables, we must still verify that the states are well defined on the completion.

Fortunately, we have, by definition (1.69),

$$\forall \omega \in \Omega : \forall A \in \mathcal{O}_+ : |\omega(A)| \leq \|A\|, \quad (1.84)$$

i. e. all ω are continuous w. r. t. to the norm $\|\cdot\|$ and can be extended to the corresponding norm completion \mathcal{O} .

In order to obtain an algebra structure, we still need to define a multiplication. Note that in the case of operators on a Hilbert space, a straightforward multiplication will not work, because the set self-adjoint operators is not closed under multiplication, unless these operators commute

$$(AB)^* = B^*A^* = BA \neq AB.$$

Nevertheless, we can construct a symmetrical product from the sums of observables and their powers,

$$A \circ B = \frac{1}{2} ((A + B)^2 - A^2 - B^2) = B \circ A \in \mathcal{O}, \quad (1.85)$$

which is unfortunately neither guaranteed to be distributive or associative. To proceed, we shall now make the mild technical assumption that the product is homogeneous, i. e.

$$\forall A, B \in \mathcal{O}, \lambda \in \mathbf{R} : A \circ (\lambda B) = \lambda(A \circ B) = (\lambda A) \circ B. \quad (1.86)$$

The motivation for this assumption is that if \mathcal{O} was an associative algebra, we would have

$$A \circ B = \frac{1}{2} (AB + BA)$$

which satisfies (1.86), of course.

Lemma 1.17. *The homogeneity (1.86) implies that (1.85) is distributive.*

Proof. From

$$(A + B)^2 = A^2 + B^2 + 2A \circ B \quad (1.87a)$$

$$(A - B)^2 = A^2 + B^2 + 2A \circ (-B) = A^2 + B^2 - 2A \circ B \quad (1.87b)$$

we find

$$A \circ B = \frac{1}{4} ((A + B)^2 - (A - B)^2) \quad (1.88a)$$

$$A^2 + B^2 = \frac{1}{2} ((A + B)^2 + (A - B)^2) . \quad (1.88b)$$

Then

$$\begin{aligned} & 2(A + B) \circ C - 2A \circ C - 2B \circ C \stackrel{(1.85)}{=} \\ & (A + B + C)^2 - (A + B)^2 - C^2 - (A + C)^2 + A^2 + C^2 - (B + C)^2 + B^2 + C^2 \\ & = ((A + B + C)^2 + A^2) + (B^2 + C^2) - ((A + B)^2 + (A + C)^2) - (B + C)^2 \\ & \stackrel{(1.88b)}{=} \frac{1}{2} \left((2A + B + C)^2 + (B + C)^2 + (B + C)^2 + (B - C)^2 \right. \\ & \quad \left. - (2A + B + C)^2 - (B - C)^2 \right) - (B + C)^2 = 0 \quad (1.89) \end{aligned}$$

i. e.

$$(A + B) \circ C = A \circ C + B \circ C \quad (1.90a)$$

and by symmetry

$$C \circ (A + B) = C \circ A + C \circ B . \quad (1.90b)$$

□

From the distributivity, symmetry and homogeneity of \circ and the linearity and positivity of ω follows

$$\begin{aligned} & \forall \omega \in \Omega : \forall \lambda \in \mathbf{R} : 0 \leq \omega((A + \lambda B)^2) \\ & = \omega((A + \lambda B) \circ (A + \lambda B)) = \omega(A^2) + \lambda^2 \omega(B^2) + 2\lambda \omega(A \circ B) \quad (1.91) \end{aligned}$$

and using the same argument as in the proof of the CSI (1.48b), we find

$$|\omega(A \circ B)| \leq \sqrt{\omega(A^2)} \sqrt{\omega(B^2)} \quad (1.92)$$

and therefore

$$\|A \circ B\| \leq \sqrt{\|A^2\|} \sqrt{\|B^2\|} = \|A\| \|B\| . \quad (1.93)$$

From the estimate³

$$\begin{aligned} \|A^2 - B^2\| &= \|(A - B) \circ (A + B)\| \leq \|A + B\| \|A - B\| \\ &\leq (\|A - B\| + 2\|B\|) \|A - B\| \end{aligned} \quad (1.94)$$

we see that the square $A \mapsto A^2$ is continuous in the norm topology

$$\lim_{A \rightarrow B} A^2 = B^2. \quad (1.95a)$$

Furthermore

$$\|A^2 - B^2\| \leq \max\{\|A\|^2, \|B\|^2\} \quad (1.95b)$$

and we see that \mathcal{O} forms a so-called *Segal system* [Seg47]. It was been shown by Segal [Seg47], that Segal systems have enough structure to recover most of quantum mechanics.

However, since the mathematical structure can become quite involved, we shall make another *technical assumption*, namely that there is a complex⁴ algebra extension $(\mathcal{A}, *)$ of \mathcal{O} with an associative, but not necessarily commutative, product, such that $\forall A, B \in \mathcal{O}$ and $\lambda, \mu \in \mathbf{C}$

$$A \circ B = \frac{1}{2}(AB + BA) \quad (1.96a)$$

$$(\lambda A + \mu B)^* = \bar{\lambda}A^* + \bar{\mu}B^* \quad (1.96b)$$

$$(AB)^* = B^*A^* \quad (1.96c)$$

$$\omega(A^*A) \geq 0 \quad (1.96d)$$

$$\|AB\| = \sup_{\omega \in \Omega} |\omega(AB)| \leq \|A\| \|B\| \quad (1.96e)$$

$$\|A^*A\| = \|A^*\| \|A\| \quad (1.96f)$$

where the states ω have been extended by linearity from \mathcal{O} to \mathcal{A} and A^*A is positive for all $A \in \mathcal{A}$.

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As before on page 11, we can then infer from positivity

$$\forall \lambda \in \mathbf{C} : \omega((\lambda A + \mathbf{1})^*(\lambda A + \mathbf{1})) \geq 0 \quad (1.97)$$

³NB: $4(A - B) \circ (A + B) = (2A)^2 - (2B)^2$.

⁴Note that it is not possible to have a non-abelian associative algebra of only self-adjoint elements, because

$$(AB)^* = B^*A^* = BA \neq AB.$$

that

$$\omega(A^*) = \overline{\omega(A)} \quad (1.98a)$$

$$\|A^*\| = \|A\|. \quad (1.98b)$$

Altogether, \mathcal{A} is a C^* -algebra with identity $\mathbf{1}$, which is generated by the subset of self-adjoint elements $\mathcal{O} \subset \mathcal{A}$.

Definition 1.18 (Physical System (von Neumann)).

1. A physical system is defined by the unital C^* -Algebra \mathcal{A} generated by its observables.
2. The states of a system are normalized, positive linear functionals $\omega : \mathcal{A} \rightarrow \mathbf{C}$ on the observables. We assume the set $\mathcal{S} \subseteq \Omega$ of *physical* states to be *full*, i. e. to *separate* the observables. Vice versa, the observables are assumed to separate the states.

Note that we allow the set \mathcal{S} of physical states to be smaller than the set Ω of all normalized, positive linear functionals on \mathcal{A} .

—2—
 C^* -ALGEBRAS

2.1 Adjoining a Unit

In the following we will study the spectrum of elements of C^* -algebras. In order to define it, we need an identity element. We start by showing that such an element can always be added without loss of generality.

Theorem 2.1. *Let \mathcal{A} be a C^* -algebra without identity and $\bar{\mathcal{A}}$ denote the set of pairs*

$$\bar{\mathcal{A}} = \{(\alpha, A) : \alpha \in \mathbf{C}, A \in \mathcal{A}\} . \quad (2.1)$$

The operations

$$\mu(\alpha, A) + \lambda(\beta, B) = (\mu\alpha + \lambda\beta, \mu A + \lambda B) \quad (2.2a)$$

$$(\alpha, A)(\beta, B) = (\alpha\beta, \alpha B + \beta A + AB) \quad (2.2b)$$

$$(\alpha, A)^* = (\bar{\alpha}, A^*) \quad (2.2c)$$

turn $\bar{\mathcal{A}}$ into a $$ -algebra with identity $(1, 0)$. Then*

$$\|(\alpha, A)\|_{\bar{\mathcal{A}}} = \sup_{B \in \mathcal{A}, \|B\|=1} \|\alpha B + AB\|_{\mathcal{A}} \quad (2.3)$$

defines a norm that turns $\bar{\mathcal{A}}$ into a C^ -algebra. \mathcal{A} can be identified with the subalgebra $\{(0, A) : A \in \mathcal{A}\}$ of $\bar{\mathcal{A}}$.*

Proof. The unital $*$ -algebra properties are obvious. The triangle and product inequalities

$$\|(\alpha, A) + (\beta, B)\| \leq \|(\alpha, A)\| + \|(\beta, B)\| \quad (2.4a)$$

$$\|(\alpha, A)(\beta, B)\| \leq \|(\alpha, A)\| \|(\beta, B)\| \quad (2.4b)$$

are left as exercise. To show that $\|(\alpha, A)\| = 0$ implies $(\alpha, A) = (0, 0)$, we start with observing that

$$\|(0, A)\| = \sup_{B \in \mathcal{A}, \|B\|=1} \|AB\| . \quad (2.5)$$

On one hand, we have

$$\sup_{B \in \mathcal{A}, \|B\|=1} \|AB\| \leq \sup_{B \in \mathcal{A}, \|B\|=1} \|A\| \|B\| = \|A\|. \quad (2.6)$$

from $\|AB\| \leq \|A\| \|B\|$, while on the other

$$\sup_{B \in \mathcal{A}, \|B\|=1} \|AB\| \geq \left\| A \frac{A^*}{\|A\|} \right\| = \frac{\|AA^*\|}{\|A\|} = \|A\| \quad (2.7)$$

from the C^* -condition (1.31). Therefore

$$\|(0, A)\|_{\bar{\mathcal{A}}} = \|A\|_{\mathcal{A}} \quad (2.8)$$

and

$$\|(0, A)\| = 0 \Rightarrow \|A\| = 0 \Rightarrow A = 0. \quad (2.9)$$

Thus we only have to study the case $\alpha \neq 0$. By linearity, we can choose Without Loss Of Generality (WLOG) $\alpha = 1$. From

$$\|B - AB\| = \|(0, B)(1, -A)\| \leq \|(0, B)\| \|(1, -A)\| \quad (2.10)$$

we can infer from $\|(1, -A)\| = 0$

$$\forall B \in \mathcal{A} : \|B - AB\| = 0 \quad (2.11)$$

or

$$\forall B \in \mathcal{A} : B = AB \quad (2.12)$$

and by involution

$$\forall B \in \mathcal{A} : B = BA^*. \quad (2.13)$$

In particular with $B = A$ and $B = A^*$

$$A = AA^* = A^* \quad (2.14)$$

and then

$$B = AB = BA = B. \quad (2.15)$$

But this means that A is an identity in \mathcal{A} which is a contradiction. Thus

$$\forall 0 \neq A \in \mathcal{A} : \|(1, -A)\| > 0. \quad (2.16)$$

Finally to prove the C^* -property for $\bar{\mathcal{A}}$, we start from the one for \mathcal{A}

$$\begin{aligned} \|\alpha B + AB\|^2 &= \|(\alpha B + AB)^*(\alpha B + AB)\| \\ &= \|B^*(\bar{\alpha}\alpha B + (\alpha A^* + \bar{\alpha}A + A^*A)B)\| \end{aligned}$$

$$\leq \|B^*\| \|\bar{\alpha}\alpha B + (\alpha A^* + \bar{\alpha}A + A^*A)B\| \quad (2.17)$$

to show that

$$\begin{aligned} \|(\alpha, A)\|^2 &= \sup_{B \in \mathcal{A}, \|B\|=1} \|\alpha B + AB\|^2 \\ &\leq \sup_{B \in \mathcal{A}, \|B\|=1} \|\bar{\alpha}\alpha B + (\alpha A^* + \bar{\alpha}A + A^*A)B\| = \|(\bar{\alpha}\alpha, \alpha A^* + \bar{\alpha}A + A^*A)\| \\ &= \|(\alpha, A)^*(\alpha, A)\| \leq \|(\alpha, A)^*\| \|(\alpha, A)\| \quad (2.18) \end{aligned}$$

i. e.

$$\|(\alpha, A)\| \leq \|(\alpha, A)^*\|. \quad (2.19)$$

The same argument for $(\alpha, A)^*$ instead of (α, A) yields

$$\|(\alpha, A)^*\| \leq \|(\alpha, A)\| \quad (2.20)$$

and we have shown

$$\|(\alpha, A)^*\| = \|(\alpha, A)\|. \quad (2.21)$$

Thus we find the desired C^* -property

$$\|(\alpha, A)\|^2 \leq \|(\alpha, A)^*(\alpha, A)\| \leq \|(\alpha, A)^*\| \|(\alpha, A)\| = \|(\alpha, A)\|^2 \quad (2.22)$$

i. e.

$$\|(\alpha, A)^*(\alpha, A)\| = \|(\alpha, A)\|^2. \quad (2.23)$$

Finally, the completeness of $\bar{\mathcal{A}} = \mathbf{C} \times \mathcal{A}$ is obvious¹, since both factors are complete. \square

Definition 2.2 (adjoining an identity). The unital algebra $\bar{\mathcal{A}}$ obtained from an algebra \mathcal{A} without identity as described in theorem 2.1 will be called obtained by *adjoining an identity 1* to \mathcal{A} . We will also write

$$\bar{\mathcal{A}} = \mathbf{C}\mathbf{1} + \mathcal{A} \quad (2.24)$$

and write $\alpha\mathbf{1} + A$ for the pairs (α, A) .

¹See, e. g. [Put18], p. 35.

2.2 Ideals and Factors

Definition 2.3. A subspace $\mathcal{B} \subseteq \mathcal{A}$ is called a *left ideal*, if $\forall A \in \mathcal{A}, B \in \mathcal{B} : AB \in \mathcal{B}$. A subspace $\mathcal{B} \subseteq \mathcal{A}$ is called a *right ideal*, if $\forall A \in \mathcal{A}, B \in \mathcal{B} : BA \in \mathcal{B}$. If \mathcal{B} is both a left and a right ideal it is called a *two sided ideal*.

Remark 2.4. Every ideal is a (sub-)algebra

Proof. (\rightarrow exercise). □

Remark 2.5. If \mathcal{B} is self adjoint and a left or right ideal it is two sided.

Proof. (\rightarrow exercise). □

Remark 2.6. If \mathcal{I} is a two sided ideal of an algebra \mathcal{A} , the factor space \mathcal{A}/\mathcal{I} is also an algebra. This is also true for *-algebras iff \mathcal{I} is self adjoint and Banach algebras iff \mathcal{I} is complete.

Proof. (\rightarrow exercise). □

Caveat: a two sided ideal in a *-algebra is *not* necessarily self adjoint. This can be seen from Stefan Waldmann's "universal counterexample" of $\mathcal{A} = C(S^2)$ with $(fg)(x) = f(x)g(x)$ and $f^*(x) = \bar{f}(-x)$. Then $\mathcal{I} = \{f \in \mathcal{A} : f(\text{north pole}) = 0\}$ is obviously a two sided ideal, but $\mathcal{I}^* = \{f \in \mathcal{A} : f(\text{south pole}) = 0\} \neq \mathcal{I}$.

2.3 Spectral Analysis

Definition 2.7 (resolvent, spectrum). Let \mathcal{A} be a unital algebra. The *resolvent set* $r_{\mathcal{A}}(A) \subset \mathbf{C}$ is the set of $\lambda \in \mathbf{C}$ such that $\lambda \mathbf{1} - A$ is *invertible*, i. e. has a two-sided inverse. The *spectrum* $\sigma_{\mathcal{A}}(A) = \mathbf{C} \setminus r_{\mathcal{A}}(A)$ is the complement of the resolvent set. The map

$$\begin{aligned} R : r_{\mathcal{A}}(A) &\rightarrow \mathcal{A} \\ \lambda &\mapsto R(\lambda) = (\lambda \mathbf{1} - A)^{-1} \end{aligned} \tag{2.25}$$

is called the *resolvent* of A .

Definition 2.8. If $\bar{\mathcal{A}}$ is the algebra obtained from an algebra \mathcal{A} without identity by adjoining an identity. We *define* for all $A \in \mathcal{A} \subset \bar{\mathcal{A}}$

$$r_{\mathcal{A}}(A) = r_{\bar{\mathcal{A}}}(A) \tag{2.26a}$$

$$\sigma_{\mathcal{A}}(A) = \sigma_{\bar{\mathcal{A}}}(A). \tag{2.26b}$$

The most straightforward approach to spectral analysis uses series expansion and analytic continuation. Formally, we can construct the resolvent for $\lambda \neq 0$ using the geometric series

$$\frac{1}{\lambda \mathbf{1} - A} = \frac{1}{\lambda} \frac{1}{\mathbf{1} - A/\lambda} = \frac{1}{\lambda} \sum_{m=0}^{\infty} \left(\frac{A}{\lambda}\right)^m. \quad (2.27)$$

Indeed, since $\|A^n\| \leq \|A\|^n$, the series converges absolutely in the norm topology in a Banach algebra if $|\lambda| > \|A\|$ and we may reorder the series (2.27)

$$(\lambda \mathbf{1} - A) \frac{1}{\lambda} \sum_{m=0}^{\infty} \left(\frac{A}{\lambda}\right)^m = \sum_{m=0}^{\infty} \left(\frac{A}{\lambda}\right)^m - \sum_{m=0}^{\infty} \left(\frac{A}{\lambda}\right)^{m+1} = \left(\frac{A}{\lambda}\right)^0 = \mathbf{1}. \quad (2.28)$$

Therefore the spectrum is bounded

$$\sigma_{\mathcal{A}}(A) \subseteq \{\lambda \in \mathbf{C} : |\lambda| \leq \|A\|\}. \quad (2.29)$$

Analogously, we can write formally for all $\lambda_0 \in r_{\mathcal{A}}(A)$

$$\begin{aligned} \frac{1}{\lambda \mathbf{1} - A} &= \frac{1}{\lambda_0 \mathbf{1} - A - (\lambda_0 - \lambda) \mathbf{1}} = \frac{1}{\lambda_0 \mathbf{1} - A} \frac{1}{\mathbf{1} - \left(\frac{\lambda_0 - \lambda}{\lambda_0 \mathbf{1} - A}\right)} \\ &= \frac{1}{\lambda_0 \mathbf{1} - A} \sum_{m=0}^{\infty} \left(\frac{\lambda_0 - \lambda}{\lambda_0 \mathbf{1} - A}\right)^m = \sum_{m=0}^{\infty} (\lambda_0 - \lambda)^m (\lambda_0 \mathbf{1} - A)^{-m-1} \end{aligned} \quad (2.30)$$

which converges absolutely for $|\lambda - \lambda_0| < \|(\lambda_0 \mathbf{1} - A)^{-1}\|^{-1}$. This establishes that for every $\lambda_0 \in r_{\mathcal{A}}(A)$, there is an open neighborhood of λ_0 that is contained in $r_{\mathcal{A}}(A)$. Thus $r_{\mathcal{A}}(A)$ is open and consequently $\sigma_{\mathcal{A}}(A)$ is closed and therefore compact, because it is bounded. These considerations suggest the

Definition 2.9 (spectral radius). The *spectral radius* of an element A of a unital Banach algebra \mathcal{A} is defined as

$$\rho(A) = \sup_{\lambda \in \sigma_{\mathcal{A}}(A)} |\lambda|. \quad (2.31)$$

As seen above, the spectral radius $\rho(A)$ is the radius of convergence of the series (2.27) for the resolvent $1/\lambda \mapsto (\lambda \mathbf{1} - A)^{-1}$ as a function of $1/\lambda$.

Lecture 05: Wed, 31. 10. 2018

In order to be able to make more detailed statements, we can prove a variation of the Cauchy-Hadamard theorem from complex analysis for power series with coefficients from a Banach algebra

Theorem 2.10. *Given a series $\{A_n\}_{n \in \mathbf{N}}$ of elements of a Banach space, the radius of convergence of the power series*

$$f(z) = \sum_{n=0}^{\infty} A_n (z - a)^n \quad (2.32)$$

is

$$r = \frac{1}{\limsup_{n \rightarrow \infty} \|A_n\|^{1/n}}. \quad (2.33)$$

Proof. **WLOG** let $a = 0$ and define $t = \limsup_{n \rightarrow \infty} \|A_n\|^{1/n}$.

1. $|z| < r$: For any $\epsilon > 0$, there are only a finite number of n such that $\|A_n\|^{1/n} \geq t + \epsilon$. Thus $\|A_n\| \leq (t + \epsilon)^n$ for all but a finite number of n and the series $\sum_n A_n z^n$ converges if $|z| < 1/(t + \epsilon) < r$
2. $|z| > r$: For any $\epsilon > 0$, there are an infinite number of n such that $\|A_n\| \geq (t - \epsilon)^n$ and the series can not converge for $|z| \geq 1/(t - \epsilon) > r$, because $A_n z^n \not\rightarrow 0$.

□

Theorem 2.11 (spectral radius). *If \mathcal{A} is a unital Banach algebra, then the spectral radius of $A \in \mathcal{A}$ is*

$$\rho(A) = \inf_{n \in \mathbf{N}} \|A^n\|^{1/n} = \lim_{n \rightarrow \infty} \|A^n\|^{1/n} \leq \|A\|. \quad (2.34)$$

The limit exists and $\sigma_{\mathcal{A}}(A)$ is a non-empty compact subset of \mathbf{C} .

Proof. Defining

$$r = \inf_{n \in \mathbf{N}} \|A^n\|^{1/n}, \quad (2.35)$$

we have obviously

$$\forall n \in \mathbf{N} : r \leq \|A^n\|^{1/n} \leq \|A\| \quad (2.36)$$

and therefore

$$r \leq \liminf_{n \rightarrow \infty} \|A^n\|^{1/n}. \quad (2.37)$$

For all $\epsilon > 0$, from (2.35) there must be an m such that $\|A^m\|^{1/m} < r + \epsilon$.
Using

$$\forall n \in \mathbf{N} : \exists k_n, l_n \in \mathbf{N} : n = mk_n + l_n \wedge 0 \leq l_n < m \quad (2.38)$$

with²

$$\lim_{n \rightarrow \infty} \frac{mk_n}{n} = 1 \quad (2.39a)$$

$$\lim_{n \rightarrow \infty} \frac{l_n}{n} = 0, \quad (2.39b)$$

we have

$$\|A^n\|^{1/n} = \|A^{k_n m} A^{l_n}\|^{1/n} \leq \|A^m\|^{k_n/n} \|A\|^{l_n/n} \leq (r + \epsilon)^{mk_n/n} \|A\|^{l_n/n}. \quad (2.40)$$

Therefore

$$\forall \epsilon > 0 : \limsup_{n \rightarrow \infty} \|A^n\|^{1/n} \leq r + \epsilon \quad (2.41)$$

and since ϵ can be arbitrarily small, the limit exists

$$\limsup_{n \rightarrow \infty} \|A^n\|^{1/n} = \liminf_{n \rightarrow \infty} \|A^n\|^{1/n} = \lim_{n \rightarrow \infty} \|A^n\|^{1/n} = r. \quad (2.42)$$

On The Other Hand (OTOH), using theorem 2.10, we see that r is the radius of convergence of the series (2.27) in $1/\lambda$ for the resolvent. Thus $\rho(A) = r$.

We have already seen above that $\sigma_{\mathcal{A}}(A)$ is compact. Finally, if $\sigma_{\mathcal{A}}(A)$ was empty, then $\lambda \mapsto (\lambda \mathbf{1} - A)^{-1}$ would be a holomorphic on the whole complex plane vanishing for $|\lambda| \rightarrow \infty$. Such a function vanishes everywhere and can not be the inverse of $\lambda \mathbf{1} - A$. \square

Lemma 2.12. *Let $A, B \in \mathcal{A}$ with $AB = BA$, then $(AB)^{-1} \in \mathcal{A}$ exists iff both $A^{-1} \in \mathcal{A}$ and $B^{-1} \in \mathcal{A}$ exist. Furthermore*

$$(AB)^{-1} = A^{-1}B^{-1} = B^{-1}A^{-1}. \quad (2.43)$$

Proof. By explicit construction:

- Assume that both A^{-1} and B^{-1} exist. Then

$$(AB)(A^{-1}B^{-1}) = BAA^{-1}B^{-1} = BB^{-1} = \mathbf{1} \quad (2.44a)$$

$$(AB)(B^{-1}A^{-1}) = AA^{-1} = \mathbf{1} \quad (2.44b)$$

$$(A^{-1}B^{-1})(AB) = A^{-1}B^{-1}BA = A^{-1}A = \mathbf{1} \quad (2.44c)$$

$$(B^{-1}A^{-1})(AB) = B^{-1}A^{-1}AB = B^{-1}B = \mathbf{1} \quad (2.44d)$$

²Think $k_n \approx n/m \dots$

- Assume that $(AB)^{-1}$ exists. Then

$$B^{-1} = (AB)^{-1}A = A(AB)^{-1} \quad (2.45a)$$

$$A^{-1} = (AB)^{-1}B = B(AB)^{-1}, \quad (2.45b)$$

because

$$B(AB)^{-1} = (AB)^{-1}ABB(AB)^{-1} = (AB)^{-1}BAB(AB)^{-1} = (AB)^{-1}B \quad (2.46)$$

(analogously $A(AB)^{-1} = (AB)^{-1}A$) and therefore

$$B((AB)^{-1}A) = BA(AB)^{-1} = AB(AB)^{-1} = \mathbf{1} \quad (2.47a)$$

$$((AB)^{-1}A)B = \mathbf{1} \quad (2.47b)$$

$$A((AB)^{-1}B) = AB(AB)^{-1} = \mathbf{1} \quad (2.47c)$$

$$((AB)^{-1}B)A = (AB)^{-1}AB = \mathbf{1}. \quad (2.47d)$$

□

Remark 2.13. Let \mathcal{A} be a unital algebra. Then

$$\forall A \in \mathcal{A} : (\sigma_{\mathcal{A}}(A))^n \subseteq \sigma_{\mathcal{A}}(A^n). \quad (2.48)$$

Proof. From

$$\lambda^n \mathbf{1} - A^n = (\lambda \mathbf{1} - A) (\lambda^{n-1} \mathbf{1} + \lambda^{n-2} A + \dots + A^{n-1}) \quad (2.49)$$

and lemma 2.12, we see that if $\lambda^n \mathbf{1} - A^n$ is invertible, then $\lambda \mathbf{1} - A$ must be invertible as well. In other words, if $\lambda^n \in r_{\mathcal{A}}(A^n)$, then $\lambda \in r_{\mathcal{A}}(A)$ or $(r_{\mathcal{A}}(A))^n \supseteq r_{\mathcal{A}}(A^n)$. □

Remark 2.14. Let \mathcal{A} be a unital $*$ -algebra. Then for all $A, B \in \mathcal{A}$

$$\forall \mu \in \mathbf{C} : \sigma_{\mathcal{A}}(A + \mu \mathbf{1}) = \sigma_{\mathcal{A}}(A) + \mu \quad (2.50a)$$

$$\sigma_{\mathcal{A}}(A^*) = \overline{\sigma_{\mathcal{A}}(A)} \quad (2.50b)$$

$$\sigma_{\mathcal{A}}(AB) \cup \{0\} = \sigma_{\mathcal{A}}(BA) \cup \{0\}. \quad (2.50c)$$

If A is invertible

$$\sigma_{\mathcal{A}}(A^{-1}) = (\sigma_{\mathcal{A}}(A))^{-1}. \quad (2.50d)$$

Proof. The first statement is obvious from

$$(\lambda + \mu)\mathbf{1} - (A + \mu\mathbf{1}) = \lambda\mathbf{1} - A. \quad (2.51)$$

The second statement follows from

$$\lambda\mathbf{1} - A^* = (\bar{\lambda}\mathbf{1} - A)^* \quad (2.52)$$

and

$$\forall A \in \mathcal{A} : (A^*)^{-1} = (A^{-1})^* \quad (2.53)$$

since $\mathbf{1} = AB = BA \Leftrightarrow \mathbf{1} = A^*B^* = B^*A^*$. If $\lambda \in r_{\mathcal{A}}(BA)$, then

$$(\lambda\mathbf{1} - AB)A \frac{1}{\lambda\mathbf{1} - BA} = A$$

from $(\lambda\mathbf{1} - AB)A = A(\lambda\mathbf{1} - BA)$. Multiplying from the right by B

$$(\lambda\mathbf{1} - AB)A \frac{1}{\lambda\mathbf{1} - BA} B = AB$$

and adding $\lambda\mathbf{1} - AB$ gives

$$(\lambda\mathbf{1} - AB) + (\lambda\mathbf{1} - AB)A \frac{1}{\lambda\mathbf{1} - BA} B = \lambda\mathbf{1}.$$

i. e.

$$(\lambda\mathbf{1} - AB) \left(\mathbf{1} + A \frac{1}{\lambda\mathbf{1} - BA} B \right) = \lambda\mathbf{1} \quad (2.54)$$

and therefore $\lambda \in r_{\mathcal{A}}(AB)$, as long as $\lambda \neq 0$, since we have explicitly computed an inverse for $\lambda\mathbf{1} - AB$. Including the case $\lambda = 0$, we obtain the third statement. Finally, if A is invertible, we know that $0 \notin \sigma_{\mathcal{A}}(A)$. Thus, **WLOG** $\lambda \neq 0$ and

$$\lambda\mathbf{1} - A = \lambda A (A^{-1} - \lambda^{-1}\mathbf{1}) \quad (2.55a)$$

$$\lambda^{-1}\mathbf{1} - A^{-1} = \lambda^{-1}A^{-1} (A - \lambda\mathbf{1}). \quad (2.55b)$$

From the first equation $\lambda \in \sigma_{\mathcal{A}}(A) \Rightarrow \lambda^{-1} \in \sigma_{\mathcal{A}}(A^{-1})$ and from the second the reverse direction, i. e. the fourth statement. \square

Theorem 2.15. *Let \mathcal{A} be a unital C^* -algebra.*

1. *If A is normal then*

$$\rho(A) = \|A\|. \quad (2.56)$$

2. If A is an isometry, i. e. $A^*A = \mathbf{1}$, but not necessarily $AA^* = \mathbf{1}$, then

$$\rho(A) = 1. \quad (2.57)$$

3. If A is unitary then

$$\sigma_{\mathcal{A}}(A) \subseteq \{\lambda \in \mathbf{C} : |\lambda| = 1\}. \quad (2.58)$$

4. If A is self-adjoint then

$$\sigma_{\mathcal{A}}(A) \subseteq [-\|A\|, \|A\|] \subset \mathbf{R} \quad (2.59a)$$

$$\sigma_{\mathcal{A}}(A^2) \subseteq [0, \|A^2\|] \subset \mathbf{R}. \quad (2.59b)$$

5. For all $A \in \mathcal{A}$ and general polynomials P

$$\sigma_{\mathcal{A}}(P(A)) = P(\sigma_{\mathcal{A}}(A)). \quad (2.60)$$

Proof. 1. From

$$\begin{aligned} \|A^{2^n}\|^2 &\stackrel{C^*\text{-prop.}}{=} \|(A^*)^{2^n} A^{2^n}\| \stackrel{A^*A=AA^*}{=} \|(A^*A)^{2^n}\| \\ &\stackrel{C^*\text{-prop.}}{=} \|(A^*A)^{2^{n-1}}\|^2 = \dots = \|A^*A\|^{2^n} \stackrel{C^*\text{-prop.}}{=} \|A\|^{2^{n+1}} \end{aligned} \quad (2.61)$$

we find

$$\rho(A) = \lim_{n \rightarrow \infty} \|A^{2^n}\|^{2^{-n}} = \|A\|. \quad (2.62)$$

2. From

$$\|A^n\|^2 \stackrel{C^*\text{-prop.}}{=} \|(A^*)^n A^n\| = \|(A^*)^{n-1} A^{n-1}\| = \dots = \|\mathbf{1}\| = 1 \quad (2.63)$$

we find

$$\rho(A) = \lim_{n \rightarrow \infty} \|A^n\|^{-n} = 1. \quad (2.64)$$

3. Every unitary A is isometric. Therefore $\sigma_{\mathcal{A}}(A)$ is contained in the unit disc. Moreover

$$\sigma_{\mathcal{A}}(A) = \overline{\sigma_{\mathcal{A}}(A^*)} = \overline{\sigma_{\mathcal{A}}(A^{-1})} = \left(\overline{\sigma_{\mathcal{A}}(A)}\right)^{-1} \quad (2.65)$$

and the spectrum must lie on the boundary of the unit disk, i. e. on the unit circle.

Lecture 06: Fri, 02. 11. 2018

4. Every self-adjoint A is normal and therefore $\rho(A) = \|A\|$. If $|1/\lambda| > \|A\|$, then $1/\lambda \in r_{\mathcal{A}}(A)$ and $\mathbf{1} + i|\lambda|A$ is invertible. Thus

$$U = (\mathbf{1} - i|\lambda|A) \frac{1}{\mathbf{1} + i|\lambda|A} \quad (2.66)$$

exists and is unitary

$$\begin{aligned} U^*U &= \frac{1}{\mathbf{1} - i|\lambda|A} (\mathbf{1} + i|\lambda|A) (\mathbf{1} - i|\lambda|A) \frac{1}{\mathbf{1} + i|\lambda|A} \\ &= \frac{1}{\mathbf{1} - i|\lambda|A} (\mathbf{1} - i|\lambda|A) (\mathbf{1} + i|\lambda|A) \frac{1}{\mathbf{1} + i|\lambda|A} = \mathbf{1} \end{aligned} \quad (2.67)$$

(the proof of $UU^* = \mathbf{1}$ is identical). From

$$\begin{aligned} \left| \frac{1 - i|\lambda|\alpha}{1 + i|\lambda|\alpha} \right|^2 &= \frac{1 + i|\lambda|\bar{\alpha}}{1 + i|\lambda|\alpha} \cdot \frac{1 - i|\lambda|\alpha}{1 - i|\lambda|\bar{\alpha}} = 1 \Leftrightarrow \\ 1 + |\lambda|^2|\alpha|^2 - i|\lambda|(\alpha - \bar{\alpha}) &= 1 + |\lambda|^2|\alpha|^2 + i|\lambda|(\alpha - \bar{\alpha}) \Leftrightarrow \alpha = \bar{\alpha} \end{aligned} \quad (2.68)$$

for $\lambda \neq 0$ and the previous statement, we infer that

$$\forall \alpha \neq \bar{\alpha} : \frac{1 - i|\lambda|\alpha}{1 + i|\lambda|\alpha} \in r_{\mathcal{A}}(U). \quad (2.69)$$

OTOH

$$\begin{aligned} \frac{1 - i|\lambda|\alpha}{1 + i|\lambda|\alpha} \mathbf{1} - U &= \frac{1 - i|\lambda|\alpha}{1 + i|\lambda|\alpha} \mathbf{1} - (\mathbf{1} - i|\lambda|A) \frac{1}{\mathbf{1} + i|\lambda|A} = \\ \frac{1}{1 + i|\lambda|\alpha} ((\mathbf{1} + i|\lambda|A)(1 - i|\lambda|\alpha) - (\mathbf{1} - i|\lambda|A)(1 + i|\lambda|\alpha)) &\frac{1}{\mathbf{1} + i|\lambda|A} \\ &= \frac{2i|\lambda|}{1 + i|\lambda|\alpha} (A - \alpha\mathbf{1}) \frac{1}{\mathbf{1} + i|\lambda|A} \end{aligned} \quad (2.70)$$

and $A - \alpha\mathbf{1}$ must be invertible for all $\alpha \neq \bar{\alpha}$. Thus

$$\sigma_{\mathcal{A}}(A) \subseteq \{\lambda \in \mathbf{C} : |\lambda| \leq \|A\| \wedge \lambda = \bar{\lambda}\}. \quad (2.71)$$

The positivity of the spectrum of a positive element will follow from the following statement.

5. For commuting $\{A_i\}_{i \in I \subset \mathbf{N}}$, the invertibility of a product $A_1 A_2 \cdots A_n$ is equivalent to the invertibility of all A_i by lemma 2.12. By the fundamental theorem of algebra, we can now find $\alpha, \alpha_i \in \mathbf{C}$ such that

$$P(x) - \lambda = \alpha \prod_{i=1}^n (x - \alpha_i) \quad (2.72)$$

and equivalently

$$P(A) - \lambda \mathbf{1} = \alpha \prod_{i=1}^n (A - \alpha_i \mathbf{1}) \quad (2.73)$$

since all powers of A commute. Thus the Left Hand Side (LHS) is invertible iff all factors of the Right Hand Side (RHS) are invertible

$$\lambda \in \sigma_{\mathcal{A}}(P(A)) \Leftrightarrow \exists i : \alpha_i \in \sigma_{\mathcal{A}}(A). \quad (2.74)$$

However, $\forall i : P(\alpha_i) = \lambda$ by (2.72) and therefore $\sigma_{\mathcal{A}}(P(A)) = P(\sigma_{\mathcal{A}}(A))$. \square

Remark 2.16. The *spectral mapping theorem* for polynomials $\sigma_{\mathcal{A}}(P(A)) = P(\sigma_{\mathcal{A}}(A))$ holds for all $A \in \mathcal{A}$. For normal elements one can prove a stronger statement

$$\sigma_{\mathcal{A}}(f(A)) = f(\sigma_{\mathcal{A}}(A)) \quad (2.75)$$

that holds for continuous functions f . We don't have the tools yet to prove it, but the crucial ingredient is that the $*$ -algebra generated by a $A \in \mathcal{A}$ is abelian iff $AA^* = A^*A$. In this case we can use theorem 1.14.

2.4 Projections

The resolvent $r_{\mathcal{A}}(A) \ni z \mapsto (z\mathbf{1} - A)^{-1} \in \mathcal{A}$ is a very powerful object that contains more information than just the location of the spectrum.

Given $A \in \mathcal{A}$ and a closed path $\Gamma \subset r_{\mathcal{A}}(A) \subset \mathbf{C}$, we can define an element of \mathcal{A} as

$$\mathcal{A} \ni P_{\Gamma}^A = \int_{\Gamma} \frac{dz}{2\pi i} R(z) = \int_{\Gamma} \frac{dz}{2\pi i} \frac{1}{z\mathbf{1} - A}. \quad (2.76)$$

Note that the integral in (2.76) is well defined, because the resolvent $z \mapsto (z\mathbf{1} - A)^{-1}$ is holomorphic on $r_{\mathcal{A}}(A)$. Such integrals can be constructed without measure theory not only for functions $\mathbf{C} \rightarrow \mathbf{C}$ but also for $\mathbf{C} \rightarrow B$, where B is an arbitrary banach space, cf., e. g., section 9.6 of [Die68]³

³Or the french [Die69] and German [Die85] translations.

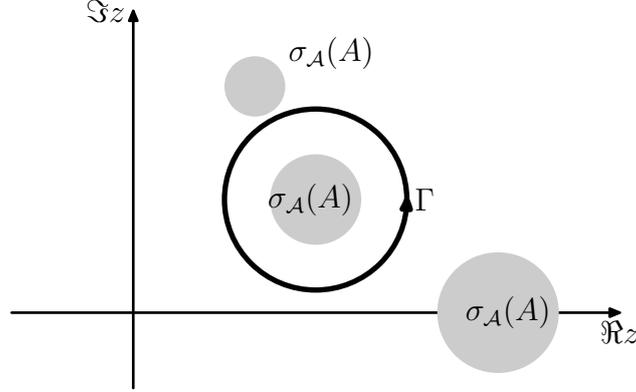


Figure 2.1: *Integration path avoiding the components of $\sigma_{\mathcal{A}}(A)$ in (2.76).*

If Γ does not encircle any part of the spectrum, the integrand $z \mapsto R(z)$ is holomorphic everywhere inside of Γ and $P_{\Gamma}^A = 0$ by Cauchy's theorem.

In the following we will need the *resolvent identity*

Remark 2.17 (Hilbert's identity).

$$\forall z_1, z_2 \in r_{\mathcal{A}}(A) : R(z_1) - R(z_2) = (z_2 - z_1)R(z_1)R(z_2). \quad (2.77)$$

Proof. It follows from multiplying

$$(R(z_2))^{-1} - (R(z_1))^{-1} = (z_2 \mathbf{1} - A) - (z_1 \mathbf{1} - A) = (z_2 - z_1) \mathbf{1} \quad (2.78)$$

by $R(z_1)$ from the left and $R(z_2)$ from the right. \square

For every pair of paths $\Gamma_{1,2}$, we can compute the product

$$\begin{aligned} P_{\Gamma_1}^A P_{\Gamma_2}^A &= \int_{\Gamma_1} \frac{dz_1}{2\pi i} \int_{\Gamma_2} \frac{dz_2}{2\pi i} \frac{1}{z_1 \mathbf{1} - A} \frac{1}{z_2 \mathbf{1} - A} \\ &\stackrel{(2.77)}{=} \int_{\Gamma_1} \frac{dz_1}{2\pi i} \int_{\Gamma_2} \frac{dz_2}{2\pi i} \frac{1}{z_2 - z_1} \left(\frac{1}{z_1 \mathbf{1} - A} - \frac{1}{z_2 \mathbf{1} - A} \right) \\ &= \int_{\Gamma_1} \frac{dz_1}{2\pi i} \frac{1}{z_1 \mathbf{1} - A} \int_{\Gamma_2} \frac{dz_2}{2\pi i} \frac{1}{z_2 - z_1} - \int_{\Gamma_2} \frac{dz_2}{2\pi i} \frac{1}{z_2 \mathbf{1} - A} \int_{\Gamma_1} \frac{dz_1}{2\pi i} \frac{1}{z_2 - z_1} \\ &= \int_{\Gamma_1} \frac{dz_1}{2\pi i} \frac{1}{z_1 \mathbf{1} - A} w_{\Gamma_2}(z_1) + \int_{\Gamma_2} \frac{dz_2}{2\pi i} \frac{1}{z_2 \mathbf{1} - A} w_{\Gamma_1}(z_2), \quad (2.79) \end{aligned}$$

where the *winding number* of a curve $\Gamma \subset \mathbf{C}$ relative to a point $z \in \mathbf{C}$

$$w_{\Gamma}(z) = \int_{\Gamma} \frac{d\zeta}{2\pi i} \frac{1}{\zeta - z} \quad (2.80)$$

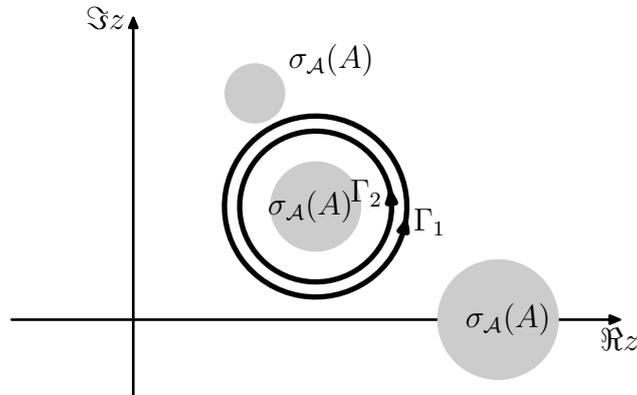


Figure 2.2: Integration paths avoiding the components of $\sigma_{\mathcal{A}}(A)$ in (2.4).

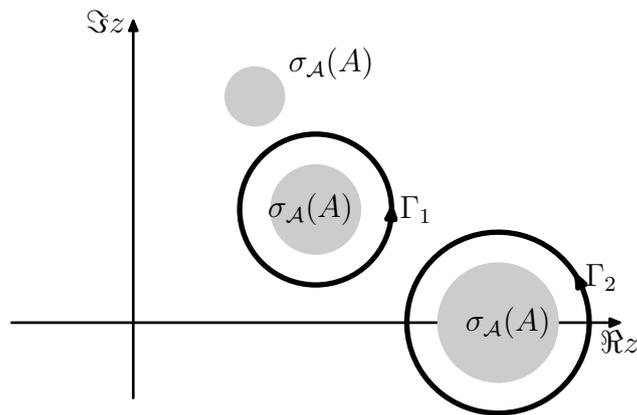


Figure 2.3: Integration paths avoiding the components of $\sigma_{\mathcal{A}}(A)$ in (2.4).

vanishes if z is outside of Γ . Thus, if we use only paths with winding number 1 and choose Γ_1 to lie completely inside Γ_2 , or vice versa, we have either $w_{\Gamma_1}(z_2) = 1 \wedge w_{\Gamma_2}(z_1) = 0$ or $w_{\Gamma_2}(z_1) = 1 \wedge w_{\Gamma_1}(z_2) = 0$ and find

$$P_{\Gamma_1}^A P_{\Gamma_2}^A = P_{\Gamma_{1,2}}^A.$$

Since we are free to choose Γ_2 as a small deformation of Γ_1 , we have derived

$$P_{\Gamma}^A P_{\Gamma}^A = P_{\Gamma}^A,$$

i. e. that P_{Γ}^A is a projection.

OTOH, if they have relative winding number 0, then $w_{\Gamma_1}(z_2) = w_{\Gamma_2}(z_1) = 0$, i. e.

$$P_{\Gamma_1}^A P_{\Gamma_2}^A = 0$$

and the projections are orthogonal. This includes the trivial cases when $P_{\Gamma_i}^A = 0$, because Γ does not encircle a part of the spectrum.

Furthermore, if *all* of $\sigma_{\mathcal{A}}(A)$ is enclosed by Γ , we can deform the contour and send it to infinity

$$\begin{aligned} P_{\Gamma}^A &= \int_{\Gamma} \frac{dz}{2\pi i} R(z) = \int_{|z|=r > \rho(A)} \frac{dz}{2\pi i} R(z) = \lim_{r \rightarrow \infty} \int_{|z|=r} \frac{dz}{2\pi i} R(z) \\ &= \sum_{n=1}^{\infty} \lim_{r \rightarrow \infty} \int_{|z|=r} \frac{dz}{2\pi i} \frac{A^{n-1}}{z^n} = \sum_{n=1}^{\infty} \delta_{n,1} A^{n-1} = \mathbf{1}. \end{aligned} \quad (2.81)$$

These considerations show that for every path Γ with winding number 1 the element

$$P_{\Gamma}^A = \int_{\Gamma} \frac{dz}{2\pi i} \frac{1}{z\mathbf{1} - A} \quad (2.82)$$

can be interpreted as a projection on the part of the spectrum enclosed by Γ .

Indeed, if z_0 is an *isolated* point in the spectrum and Γ encircles only z_0 , one can show that

$$(z_0\mathbf{1} - A)P_{\Gamma}^A = 0,$$

because $z_0\mathbf{1} - A$ “cancels the pole” of the resolvent at z_0 . However, instead of proving this special case, we will consider a more general functional calculus now.

2.5 Holomorphic Functional Calculus

For *holomorphic* $f : \mathbf{C} \rightarrow \mathbf{C}$, we can define a corresponding $\hat{f} : \mathcal{A} \rightarrow \mathcal{A}$ via

$$\hat{f}(A) = \int_{\Gamma} \frac{dz}{2\pi i} \frac{f(z)}{z\mathbf{1} - A} \quad (2.83)$$

where Γ encircles all of $\sigma_{\mathcal{A}}(A)$ with winding number one.

Lecture 07: Wed, 07.11.2018

We can compute the product of two functions, choosing Γ' on the *outside* of Γ :

$$\hat{f}_1(A)\hat{f}_2(A) = \int_{\Gamma} \frac{dz}{2\pi i} \int_{\Gamma'} \frac{dz'}{2\pi i} \frac{f_1(z)}{z\mathbf{1} - A} \frac{f_2(z')}{z'\mathbf{1} - A}$$

$$\begin{aligned}
&= \int_{\Gamma} \frac{dz}{2\pi i} \int_{\Gamma'} \frac{dz'}{2\pi i} \frac{f_1(z)f_2(z')}{z'-z} \left(\frac{1}{z\mathbf{1}-A} - \frac{1}{z'\mathbf{1}-A} \right) \\
&= \int_{\Gamma} \frac{dz}{2\pi i} \frac{f_1(z)}{z\mathbf{1}-A} \underbrace{\int_{\Gamma'} \frac{dz'}{2\pi i} \frac{f_2(z')}{z'-z}}_{=f_2(z)} - \int_{\Gamma'} \frac{dz'}{2\pi i} \frac{f_2(z')}{z'\mathbf{1}-A} \underbrace{\int_{\Gamma} \frac{dz}{2\pi i} \frac{f_1(z)}{z'-z}}_{=0} \\
&= \int_{\Gamma} \frac{dz}{2\pi i} \frac{f_1(z)}{z\mathbf{1}-A} f_2(z) = \int_{\Gamma} \frac{dz}{2\pi i} \frac{(f_1 \cdot f_2)(z)}{z\mathbf{1}-A} = \widehat{(f_1 \cdot f_2)}(A). \quad (2.84)
\end{aligned}$$

i. e. $f \mapsto \hat{f}$ is an *homomorphism*

$$\hat{f}_1(A)\hat{f}_2(A) = \widehat{(f_1 \cdot f_2)}(A). \quad (2.85)$$

In physicist's notation, an analogous functional calculus for self-adjoint operators on a Hilbert space is written

$$\hat{f}(A) = \int d\mu(a) f(a) |a\rangle\langle a| \quad (2.86)$$

with

$$\begin{aligned}
\hat{f}_1(A)\hat{f}_2(A) &= \int d\mu(a) \int d\mu(a') f_1(a)f_2(a') |a\rangle\langle a|a'\rangle\langle a'| \\
&= \int d\mu(a) (f_1 \cdot f_2)(a) |a\rangle\langle a| = \widehat{(f_1 \cdot f_2)}(A). \quad (2.87)
\end{aligned}$$

2.5.1 Examples

1. For consistency, we should have $\hat{f}(\lambda\mathbf{1}) = f(\lambda)\mathbf{1}$. Indeed

$$\hat{f}(\lambda\mathbf{1}) = \int_{\Gamma} \frac{dz}{2\pi i} \frac{f(z)}{z\mathbf{1}-\lambda\mathbf{1}} = \int_{\Gamma} \frac{dz}{2\pi i} \frac{f(z)}{z-\lambda} \mathbf{1} = f(\lambda) \mathbf{1}. \quad (2.88)$$

2. Also $\widehat{\text{id}} : A \rightarrow A$ for $\text{id} : z \rightarrow z$

$$\begin{aligned}
\widehat{\text{id}}(A) &= \int_{\Gamma} \frac{dz}{2\pi i} \frac{z}{z\mathbf{1}-A} = \mathbf{1} \underbrace{\int_{\Gamma} \frac{dz}{2\pi i}}_{=0} + A \underbrace{\int_{\Gamma} \frac{dz}{2\pi i} \frac{1}{z\mathbf{1}-A}}_{=1} = A, \quad (2.89) \\
&\quad \mathbf{1} + A \frac{1}{z\mathbf{1}-A}
\end{aligned}$$

and $\widehat{\text{sq}} : A \rightarrow A^2$ for $\text{sq} : z \rightarrow z^2$

$$\begin{aligned}
\widehat{\text{sq}}(A) &= \int_{\Gamma} \frac{dz}{2\pi i} \underbrace{\frac{z^2}{z\mathbf{1} - A}} \\
&\quad (z\mathbf{1} - A) + 2A + A^2 \frac{1}{z\mathbf{1} - A} \\
&= \underbrace{\int_{\Gamma} \frac{dz}{2\pi i} (z\mathbf{1} - A)}_{=0} + 2A \underbrace{\int_{\Gamma} \frac{dz}{2\pi i}}_{=0} + A^2 \underbrace{\int_{\Gamma} \frac{dz}{2\pi i} \frac{1}{z\mathbf{1} - A}}_{=1} = A^2. \quad (2.90)
\end{aligned}$$

Obviously, this holds analogously for higher powers and all polynomials.

2.6 Holomorphic Spectral Mapping Theorem

Theorem 2.18. *If $f : \mathbf{C} \rightarrow \mathbf{C}$ is holomorphic on a neighborhood of $\sigma_{\mathcal{A}}(A)$, then $f(\sigma_{\mathcal{A}}(A)) = \sigma_{\mathcal{A}}(\widehat{f}(A))$, with \widehat{f} from (2.83).*

Proof. For all $\lambda \in \mathbf{C}$, $f_{\lambda} : z \mapsto \lambda - f(z)$ is holomorphic on a neighborhood of $\sigma_{\mathcal{A}}(A)$.

- If $\lambda \notin f(\sigma_{\mathcal{A}}(A))$, then $h_1 : z \mapsto (f_{\lambda}(z))^{-1} = (\lambda - f(z))^{-1}$ is also holomorphic on a neighborhood of $\sigma_{\mathcal{A}}(A)$. Using (2.85) and the function $1 : z \mapsto 1$, we find

$$\begin{aligned}
(\lambda\mathbf{1} - \widehat{f}(A)) \widehat{h}_1(A) &= \widehat{f}_{\lambda}(A) \widehat{h}_1(A) = \widehat{(f_{\lambda} \cdot h_1)}(A) = \widehat{1}(A) = \mathbf{1} \\
&= \widehat{(h_1 \cdot f_{\lambda})}(A) = \widehat{h}_1(A) \widehat{f}_{\lambda}(A) = \widehat{h}_1(A) (\lambda\mathbf{1} - \widehat{f}(A)) \quad (2.91)
\end{aligned}$$

that an inverse of $\lambda\mathbf{1} - \widehat{f}(A)$ exists, i. e. $\lambda \notin \sigma_{\mathcal{A}}(\widehat{f}(A))$ and $f(\sigma_{\mathcal{A}}(A)) \supseteq \sigma_{\mathcal{A}}(\widehat{f}(A))$.

- If $\lambda \in f(\sigma_{\mathcal{A}}(A))$, then $\exists z_0 \in \sigma_{\mathcal{A}}(A)$ with $\lambda = f(z_0)$ and there is a function h_2 holomorphic on a neighborhood of $\sigma_{\mathcal{A}}(A)$ with

$$\lambda - f(z) = f(z_0) - f(z) = (z_0 - z)h_2(z). \quad (2.92)$$

Using (2.85) again, we can compute

$$\lambda\mathbf{1} - \widehat{f}(A) = (z_0\mathbf{1} - A)\widehat{h}_2(A) = \widehat{h}_2(A)(z_0\mathbf{1} - A). \quad (2.93)$$

Since $z_0\mathbf{1} - A$ is by assumption not invertible, $\lambda\mathbf{1} - \widehat{f}(A)$ can also not be invertible, i. e. $\lambda \in \sigma_{\mathcal{A}}(\widehat{f}(A))$ and $f(\sigma_{\mathcal{A}}(A)) \subseteq \sigma_{\mathcal{A}}(\widehat{f}(A))$.

The two inclusions are complementary, therefore $f(\sigma_{\mathcal{A}}(A)) = \sigma_{\mathcal{A}}(\widehat{f}(A))$. \square

Having established that f maps the resolvent set of A into the resolvent set of $\hat{f}(A)$, we can use the holomorphic functional calculus to compose maps:

Corollary 2.19.

$$\hat{f} \circ \hat{g} = \widehat{f \circ g} \quad (2.94)$$

Proof. With Γ_1 and Γ_2 chosen such that Γ_2 encircles $g(\Gamma_1)$ once, we have

$$\begin{aligned} \widehat{(f \circ g)}(A) &= \int_{\Gamma_1} \frac{dz_1}{2\pi i} \frac{(f \circ g)(z_1)}{z_1 \mathbf{1} - A} = \int_{\Gamma_1} \frac{dz_1}{2\pi i} \overbrace{\int_{\Gamma_2} \frac{dz_2}{2\pi i} \frac{f(z_2)}{z_2 - g(z_1)}}^{f(g(z_1))} \frac{1}{z_1 \mathbf{1} - A} \\ &= \int_{\Gamma_2} \frac{dz_2}{2\pi i} f(z_2) \int_{\Gamma_1} \frac{dz_1}{2\pi i} \frac{1}{z_2 - g(z_1)} \frac{1}{z_1 \mathbf{1} - A} \\ &= \int_{\Gamma_2} \frac{dz_2}{2\pi i} f(z_2) \frac{1}{z_2 \mathbf{1} - \hat{g}(A)} = \hat{f}(\hat{g}(A)) = (\hat{f} \circ \hat{g})(A), \end{aligned} \quad (2.95)$$

where we have used the holomorphic functional calculus for the family of functions

$$h_\lambda : z \mapsto \frac{1}{\lambda - z} \quad (2.96)$$

that involve only addition and pointwise inverse for which we already know from (2.85) that the holomorphic functional calculus is valid. \square

2.7 Uniqueness and Independence

As already alluded to above, it turns out that the C^* -property is very restrictive. In fact, a C^* -norm is unique and the spectrum of an element is independent of the (sub-)algebra it belongs to.

Lemma 2.20. *If a $*$ -algebra \mathcal{A} has a norm with the C^* -property, this norm is unique.*

Proof. For normal and in particular for self-adjoint elements A , we know from theorem 2.15, that the C^* -property entails

$$\|A\| = \rho(A) \quad (2.97)$$

and the latter is already determined by the algebraic structure. For general $A \in \mathcal{A}$, the C^* -property yields

$$\|A\| = \sqrt{\|A^*A\|} = \sqrt{\rho(A^*A)}. \quad (2.98)$$

\square

Given a unital subalgebra \mathcal{B} of a unital algebra \mathcal{A} , the spectrum of $A \in \mathcal{B}$ is in general a superset of the spectrum of $A \in \mathcal{A}$. Indeed,

$$r_{\mathcal{A}}(A) \supseteq r_{\mathcal{B}}(A) \quad (2.99)$$

because the inverse of $\lambda \mathbf{1} - A$ might be a element of $\mathcal{A} \setminus \mathcal{B}$. This is not the case in a C^* -algebra, however:

Theorem 2.21. *If \mathcal{B} is a unital C^* -subalgebra of a unital C^* -algebra \mathcal{A} and the unit elements of \mathcal{A} and \mathcal{B} coincide, then $\sigma_{\mathcal{A}}(A) = \sigma_{\mathcal{B}}(A)$.*

Proof. We will show that if $\lambda \mathbf{1} - A$ is invertible in \mathcal{A} , then it is invertible in the C^* -subalgebra \mathcal{C} generated by $\mathbf{1}$, A and A^* , i. e. $\sigma_{\mathcal{C}}(A) = \sigma_{\mathcal{A}}(A)$. Then

$$\sigma_{\mathcal{A}}(A) = \sigma_{\mathcal{C}}(A) = \sigma_{\mathcal{B}}(A). \quad (2.100)$$

Thus it suffices to show that if A is invertible in \mathcal{A} , then $A^{-1} \in \mathcal{C}$.

Starting with self-adjoint $A = A^*$, we know that $A - \lambda \mathbf{1}$ is invertible for all λ with a non-vanishing imaginary part. For $\lambda_0 = 2i\|A\|$, we know that the power series

$$\frac{1}{A - \lambda_0 \mathbf{1}} = -\frac{1}{\lambda_0} \sum_{n=0}^{\infty} \left(\frac{A}{\lambda_0} \right)^n \quad (2.101)$$

converges absolutely and $(A - \lambda_0 \mathbf{1})^{-1} \in \mathcal{C}$. Since $A = A^*$, the resolvent $R(\lambda) = (\lambda \mathbf{1} - A)^{-1}$ with $(R(\lambda))^* = R(\bar{\lambda})$ is normal. Furthermore $\sigma_{\mathcal{A}}((A - \lambda \mathbf{1})^{-1}) = (\sigma_{\mathcal{A}}(A) - \lambda)^{-1}$ and therefore

$$\begin{aligned} \|(A - \lambda \mathbf{1})^{-1}\| &= \rho((A - \lambda \mathbf{1})^{-1}) \\ &= \sup_{z \in \sigma_{\mathcal{A}}(A)} |(z - \lambda)^{-1}| = \left(\inf_{z \in \sigma_{\mathcal{A}}(A)} |z - \lambda| \right)^{-1} = \frac{1}{d(\lambda)}, \end{aligned} \quad (2.102)$$

where $d(\lambda)$ designates the closest distance from λ to $\sigma_{\mathcal{A}}(A)$. For purely imaginary λ , we have $d(\lambda) > |\lambda|$, because $0 \notin \sigma_{\mathcal{A}}(A)$ for invertible A and the compactness of the spectrum. Using this result, we see that

$$\begin{aligned} \frac{1}{A - \lambda \mathbf{1}} &= \frac{1}{A - \lambda_0 \mathbf{1} - (\lambda - \lambda_0) \mathbf{1}} = \frac{1}{A - \lambda_0 \mathbf{1}} \frac{1}{\mathbf{1} - \frac{\lambda - \lambda_0}{A - \lambda_0 \mathbf{1}}} \\ &= \frac{1}{A - \lambda_0 \mathbf{1}} \sum_{n=0}^{\infty} \left(\frac{\lambda - \lambda_0}{A - \lambda_0 \mathbf{1}} \right)^n \end{aligned} \quad (2.103)$$

converges for $|\lambda - \lambda_0| < d(\lambda_0) = \|(A - \lambda_0 \mathbf{1})^{-1}\|^{-1}$. For purely imaginary λ_0 , the series (2.103) will therefore converge for $\lambda = 0$ and

$$A^{-1} = \frac{1}{A - \lambda_0 \mathbf{1}} \sum_{n=0}^{\infty} \left(\frac{-\lambda_0}{A - \lambda_0 \mathbf{1}} \right)^n \in \mathcal{C}. \quad (2.104)$$

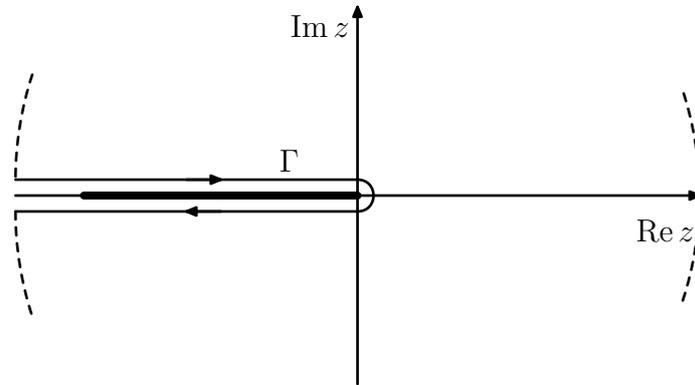


Figure 2.4: Integration contour Γ used in (2.108) to define the square root.

If A is invertible, but $A \neq A^*$, then A^*A is invertible nevertheless

$$(A^*A)^{-1} = A^{-1}(A^*)^{-1} = A^{-1}(A^{-1})^* \quad (2.105)$$

and $(A^*A)^{-1}$ lies in the C^* -algebra generated by $\mathbf{1}$ and A^*A . For

$$X = (A^*A)^{-1}A^* \in \mathcal{C} \quad (2.106)$$

we find $XA = \mathbf{1}$, i. e. $X = A^{-1}$, i. e. $A^{-1} \in \mathcal{C}$ (cf. lemma 2.12). \square

Therefore, we will henceforth write $\sigma(A)$ for $\sigma_{\mathcal{A}}(A)$.

2.8 Positive Elements

Lecture 08: Fri, 09. 11. 2018

Earlier, we have defined the positive elements A of an algebra as those that can be written as $A = B^*B$. One can equivalently characterize them by their spectrum:

Definition 2.22. The set \mathcal{A}_+ of all positive elements of a $*$ -algebra \mathcal{A} is the set of all self-adjoint elements with the spectrum $\sigma(A)$ contained in the positive real axis.

For a positive element $A \in \mathcal{A}_+$, we can construct a *square root* with

$$B = \sqrt{A} = \int_0^\infty \frac{d\lambda}{\pi} \frac{A}{\sqrt{\lambda}} \frac{1}{\lambda \mathbf{1} + A}. \quad (2.107)$$

For invertible A , this construction is compatible with the holomorphic functional calculus. In order to avoid a contribution from the circle at infinity, we construct the inverse and define $B = \sqrt{A} = A\sqrt{A^{-1}}$ later. This works indeed using the contour Γ from figure 2.4 which avoids the cut of $z \rightarrow \sqrt{z}$ and encircles $\sigma(A) \subset (0, \infty)$.

$$\begin{aligned}
\sqrt{A^{-1}} &= \int_{\Gamma} \frac{dz}{2\pi i} \frac{1}{\sqrt{z}} \frac{1}{z\mathbf{1} - A} \\
&= \lim_{\epsilon \rightarrow 0^+} \int_0^{-\infty} \frac{dx}{2\pi i} \frac{1}{\sqrt{x - i\epsilon}} \frac{1}{(x - i\epsilon)\mathbf{1} - A} + \lim_{\epsilon \rightarrow 0^+} \int_{-\infty}^0 \frac{dx}{2\pi i} \frac{1}{\sqrt{x + i\epsilon}} \frac{1}{(x + i\epsilon)\mathbf{1} - A} \\
&= \lim_{\epsilon \rightarrow 0^+} \int_0^{-\infty} \frac{dx}{2\pi i} \left(\frac{1}{\sqrt{x - i\epsilon}} \frac{1}{(x - i\epsilon)\mathbf{1} - A} - \frac{1}{\sqrt{x + i\epsilon}} \frac{1}{(x + i\epsilon)\mathbf{1} - A} \right) \\
&\stackrel{(2.110)}{=} \int_0^{-\infty} \frac{dx}{\pi} \frac{1}{\sqrt{|x|}} \frac{1}{x\mathbf{1} - A} = \int_0^{\infty} \frac{dx}{\pi} \frac{1}{\sqrt{x}} \frac{1}{x\mathbf{1} + A}, \quad (2.108)
\end{aligned}$$

where we have used

$$\lim_{\epsilon \rightarrow 0^+} \sqrt{-|x| \pm i\epsilon} = \pm i\sqrt{|x|} \quad (2.109)$$

and

$$\begin{aligned}
&\frac{1}{\sqrt{x - i\epsilon}} \frac{1}{(x - i\epsilon)\mathbf{1} - A} - \frac{1}{\sqrt{x + i\epsilon}} \frac{1}{(x + i\epsilon)\mathbf{1} - A} \\
&= \frac{i}{\sqrt{|x|}} \frac{1}{x\mathbf{1} - A - i\epsilon\mathbf{1}} + \frac{i}{\sqrt{|x|}} \frac{1}{x\mathbf{1} - A + i\epsilon\mathbf{1}} \\
&= \frac{2i}{\sqrt{|x|}} \frac{x\mathbf{1} - A}{(x\mathbf{1} - A)^2 + \epsilon^2\mathbf{1}} \xrightarrow{\epsilon \rightarrow 0^+} \frac{2i}{\sqrt{|x|}} \frac{1}{x\mathbf{1} - A}. \quad (2.110)
\end{aligned}$$

Additional material (not discussed during the lectures):

Note that (2.107) can also be constructed as a Riemannian integral. For positive A we have, from the fact that the function

$$\begin{aligned}
f : [0, \infty) &\rightarrow [0, 1) \\
z &\mapsto \frac{z}{\lambda + z}
\end{aligned} \quad (2.111)$$

is a bijection for $\lambda > 0$,

$$\sigma\left(A \frac{1}{\lambda\mathbf{1} + A}\right) = f(\sigma(A)) \subseteq f([0, \|A\|]) = \left[0, \frac{\|A\|}{\lambda + \|A\|}\right]. \quad (2.112)$$

Therefore

$$\left\|A \frac{1}{\lambda\mathbf{1} + A}\right\| = \rho\left(A \frac{1}{\lambda\mathbf{1} + A}\right) \leq \frac{\|A\|}{\lambda + \|A\|} \quad (2.113)$$

and

$$\|B\| = \left\| \int_0^\infty \frac{dx}{\pi} \frac{A}{\sqrt{x}} \frac{1}{x\mathbf{1} + A} \right\| \leq \int_0^\infty \frac{dx}{\pi} \left\| \frac{A}{\sqrt{x}} \frac{1}{x\mathbf{1} + A} \right\|$$

$$\stackrel{(2.113)}{\leq} \int_0^\infty \frac{dx}{\pi} \frac{1}{\sqrt{x}} \frac{\|A\|}{x + \|A\|} = \sqrt{\|A\|} \underbrace{\int_0^\infty \frac{dy}{\pi} \frac{1}{\sqrt{y}} \frac{1}{y + 1}}_{=1} = \sqrt{\|A\|} \quad (2.114)$$

converges.

One can show

Theorem 2.23. *A self-adjoint element A of a C^* -algebra \mathcal{A} is positive iff $\exists B \in \mathcal{A} : A = B^2 \wedge B = B^*$. If A is positive, there exists a unique positive B with $A = B^2$ and it lies in the subalgebra of \mathcal{A} generated by A .*

We will not give the proof here (see, e. g., [BR02], theorem 2.2.10), but it should be intuitively clear, that the integrand in (2.107) is positive and the corresponding Riemann sums are positive as well.

The square root plays a special role, because we can use it to define the *modulus* of an arbitrary self-adjoint $A \in \mathcal{A}$ via

$$|A| = \sqrt{A^2} \in \mathcal{A}. \quad (2.115)$$

Note that $\|A\| \in \mathbf{R}$, but $|A| \in \mathcal{A}$. One can use the modulus to write every self-adjoint $A \in \mathcal{A}$ as the difference of two positive elements:

Theorem 2.24. *The set \mathcal{A}_+ of all positive elements of a $*$ -algebra \mathcal{A} is a convex cone that is closed in the norm topology, with*

$$\mathcal{A}_+ \cap (-\mathcal{A}_+) = \{0\}. \quad (2.116)$$

If $A = A^* \in \mathcal{A}$, then

$$A_\pm = \frac{1}{2} (|A| \pm A), \quad (2.117)$$

are the unique elements of \mathcal{A} with

$$A_\pm \in \mathcal{A}_+ \quad (2.118a)$$

$$A = A_+ - A_- \quad (2.118b)$$

$$A_+ A_- = 0. \quad (2.118c)$$

Most of the proof will be skipped (see, e. g., [BR02], theorem 2.2.11). However the positivity of A_\pm , i. e. (2.118a), is intuitively clear, while (2.118b) is trivial and (2.118c) follows from

$$4A_+ A_- = (|A| + A)(|A| - A) = |A|^2 + A|A| - |A|A - A^2 = [A, |A|] = 0 \quad (2.119)$$

because $|A|$ is an element of the abelian Algebra generated by A .

Finally, one can also show the reverse direction of theorem 2.23 (see, e. g., [BR02], theorem 2.2.12)

Theorem 2.25. *If \mathcal{A} is a C^* -algebra, then*

$$\forall A \in \mathcal{A} : (A \in \mathcal{A}_+) \Leftrightarrow (\exists B \in \mathcal{A} : A = B^*B) . \quad (2.120)$$

We can use the existence of positive elements to introduce a partial order on an algebra

Definition 2.26.

$$A \geq B \Leftrightarrow A - B \geq 0 \Leftrightarrow A - B \in \mathcal{A}_+ \quad (2.121a)$$

$$A > B \Leftrightarrow A \geq B \wedge A \not\equiv B . \quad (2.121b)$$

Obviously

$$A \geq 0 \wedge A \leq 0 \Rightarrow A = 0 \quad (2.122a)$$

$$A \geq B \wedge B \geq C \Rightarrow A \geq C , \quad (2.122b)$$

but also

Theorem 2.27. $\forall A, B, C \in \mathcal{A}$, a C^* -algebra:

$$A \geq B \geq 0 \Rightarrow \|A\| \geq \|B\| \quad (2.123a)$$

$$A \geq 0 \Rightarrow A\|A\| \geq A^2 \quad (2.123b)$$

$$A \geq B \geq 0 \Rightarrow \forall C \in \mathcal{A} : C^*AC \geq C^*BC \geq 0 \quad (2.123c)$$

$$A \geq B \geq 0 \wedge \lambda > 0 \Rightarrow \frac{1}{B + \lambda \mathbf{1}} \geq \frac{1}{A + \lambda \mathbf{1}} \quad (2.123d)$$

Proof. First we observe, using $\rho(A) = \|A\|$

$$\begin{aligned} \min \sigma (\|A\| \mathbf{1} - A) &= \|A\| - \max \sigma (A) = \|A\| - \|A\| = 0 \\ &\Rightarrow \|A\| \mathbf{1} - A \geq 0 \Rightarrow B \leq A \leq \|A\| \mathbf{1} \end{aligned} \quad (2.124)$$

and then

$$\begin{aligned} 0 \leq \min \sigma (\|A\| \mathbf{1} - B) &= \|A\| - \max \sigma (B) = \|A\| - \|B\| \\ &\Rightarrow \|A\| \geq \|B\| , \end{aligned} \quad (2.125)$$

i. e. (2.123a). Similarly

$$\sigma (A - \|A\| \mathbf{1}/2) = \sigma (A) - \|A\|/2 \subseteq [-\|A\|/2, \|A\|/2] \quad (2.126)$$

and therefore

$$\sigma((A - \|A\|\mathbf{1}/2)^2) \subseteq [0, \|A\|^2/4]. \quad (2.127)$$

This means

$$0 \leq (A - \|A\|\mathbf{1}/2)^2 \leq \|A\|^2\mathbf{1}/4 \quad (2.128)$$

i. e. (2.123b)

$$A^2 - A\|A\| \leq 0. \quad (2.129)$$

We obtain (2.123c) as follows

$$\begin{aligned} A \geq B &\Rightarrow \exists D \in \mathcal{A} : A - B = D^*D \Rightarrow \\ C^*AC - C^*BC &= C^*(A - B)C = C^*D^*DC = (DC)^*(DC) \geq 0. \end{aligned} \quad (2.130)$$

And from this (2.123d)

$$\begin{aligned} A \geq B \geq 0 &\Rightarrow \forall \lambda : A + \lambda\mathbf{1} \geq B + \lambda\mathbf{1} \geq \lambda\mathbf{1} \Rightarrow \\ \forall \lambda > 0 : &\frac{1}{\sqrt{B + \lambda\mathbf{1}}}(A + \lambda\mathbf{1})\frac{1}{\sqrt{B + \lambda\mathbf{1}}} \geq \mathbf{1} \\ \Rightarrow \forall \lambda > 0 : &\sqrt{B + \lambda\mathbf{1}}\frac{1}{A + \lambda\mathbf{1}}\sqrt{B + \lambda\mathbf{1}} \leq \mathbf{1} \\ \Rightarrow \forall \lambda > 0 : &\frac{1}{A + \lambda\mathbf{1}} \leq \frac{1}{B + \lambda\mathbf{1}} \end{aligned} \quad (2.131)$$

where we have used

$$\begin{aligned} X = X^* \geq \mathbf{1} &\Rightarrow \sigma(X) = \sigma(X - \mathbf{1}) + 1 \subseteq [0, \infty) + 1 = [1, \infty) \\ &\Rightarrow \sigma(X^{-1}) \subseteq [1, \infty)^{-1} = (0, 1] \Rightarrow X^{-1} \leq \mathbf{1} \end{aligned} \quad (2.132)$$

in the next to final implication. \square

—3—

REPRESENTATIONS

Lecture 09: Wed, 14.11.2018

In order to make contact with the Hilbert space formulation of **QM**, we need to realize the algebraic formulation as operators on Hilbert space. In particular, we have to find out iff there is more than one inequivalent realization. Otherwise, the algebraic formulation will not have given us a lot of new insights.

3.1 Homomorphisms

Definition 3.1 (homomorphism of *-algebras). A **-homomorphism* π is a map

$$\begin{aligned} \pi : \mathcal{A} &\rightarrow \mathcal{B} \\ A &\mapsto \pi(A) \end{aligned} \tag{3.1}$$

that preserves the *-algebra structure, i. e. for all $\alpha, \beta \in \mathbf{C}$ and $A, B \in \mathcal{A}$

$$\pi(\alpha A + \beta B) = \alpha \pi(A) + \beta \pi(B) \tag{3.2a}$$

$$\pi(AB) = \pi(A)\pi(B) \tag{3.2b}$$

$$\pi(A^*) = (\pi(A))^* . \tag{3.2c}$$

Remark 3.2. A **-homomorphism* π preserves positivity, i. e.

$$\forall A \geq 0 : \pi(A) \geq 0 . \tag{3.3}$$

Proof. This is almost obvious:

$$\begin{aligned} \mathcal{A} \ni A \geq 0 &\Rightarrow \exists B \in \mathcal{A} : A = B^*B \\ &\Rightarrow \pi(A) = \pi(B^*)\pi(B) = (\pi(B))^* \pi(B) \geq 0 . \end{aligned} \tag{3.4}$$

□

Definition 3.3. The *kernel* of a $*$ -morphism $\pi : \mathcal{A} \rightarrow \mathcal{B}$ is defined as

$$\ker \pi = \{A \in \mathcal{A} : \pi(A) = 0\} \quad (3.5)$$

Remark 3.4. $\ker \pi$ is a two-sided $*$ -ideal.

Proof. $\forall A \in \mathcal{A}, B \in \ker \pi$:

$$\pi(AB) = \pi(A)\pi(B) = \pi(A)0 = 0 \quad (3.6a)$$

$$\pi(BA) = \pi(B)\pi(A) = 0\pi(A) = 0 \quad (3.6b)$$

$$\pi(B^*) = (\pi(B))^* = 0^* = 0 \quad (3.6c)$$

□

Therefore, $\mathcal{A}/\ker \pi$ is a $*$ -algebra and the map

Definition 3.5.

$$\begin{aligned} \hat{\pi} : \mathcal{A}_\pi = \mathcal{A}/\ker \pi &\rightarrow \mathcal{B}_\pi = \text{Im } \pi = \pi(\mathcal{A}) \\ \hat{A} &\mapsto \pi(A \in \hat{A}) \end{aligned} \quad (3.7)$$

is well defined

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\pi} & \mathcal{B}_\pi \subseteq \mathcal{B} \\ \downarrow \hat{} & \nearrow \hat{\pi} & \\ \mathcal{A}_\pi & & \end{array} \quad (3.8)$$

because

$$\pi(\hat{A}) = \pi(A + I) = \pi(A) + \pi(I) = \pi(A) + 0 = \pi(A), \quad (3.9)$$

and an isomorphism, because $\ker \hat{\pi} = \hat{0}$, by construction.

3.2 Approximate Identities and Factor Algebras

While we have shown in theorem 2.1 that one can always adjoin an identity, it is sometimes relevant that the original algebra does not contain one. However, we want to show that certain factor algebras that appear naturally, i. e. $\mathcal{A}/\ker \pi$, carry a C^* -algebra structure. In order to prove this, we need to introduce approximate identities of the ideal $\ker \pi$. In some way, approximate identities resemble the representations of Dirac δ -distributions as the limit of a sequence of functions in a given function space, while the Dirac δ -distributions themselves are not in this function space.

Definition 3.6. If \mathcal{I} is a right ideal of a C^* -algebra, an *approximate identity* of \mathcal{I} is a family¹ $\{E_\alpha\}_{\alpha \in U}$ of *positive* elements $E_\alpha \in \mathcal{I}$, indexed by a *directed set*² U with the properties

$$\|E_\alpha\| \leq 1 \quad (3.10a)$$

$$\forall \alpha \leq \beta \in U : E_\alpha \leq E_\beta \quad (3.10b)$$

$$\forall I \in \mathcal{I} : \|E_\alpha I - I\| \rightarrow 0. \quad (3.10c)$$

An approximate identity of a left ideal is defined analogously with $\|IE_\alpha - I\| \rightarrow 0$.

This definition is useful because we can show

Theorem 3.7. *Every right ideal \mathcal{I} of a C^* -algebra \mathcal{A} possesses an approximate identity.*

Proof. We can order the set \mathcal{U} of all *finite* families $\alpha = \{A_1, A_2, \dots, A_{|\alpha|}\}$ by inclusion, i. e. $\alpha \geq \beta$ **iff** β is a subfamily of α . We can define for every family α the positive algebra elements

$$\mathcal{I} \ni F_\alpha = \sum_{i=1}^{|\alpha|} A_i A_i^* \geq 0 \quad (3.11)$$

and

$$\mathcal{I} \ni E_\alpha = |\alpha| F_\alpha \frac{1}{\mathbf{1} + |\alpha| F_\alpha} = \mathbf{1} - \frac{1}{\mathbf{1} + |\alpha| F_\alpha} \leq \mathbf{1}. \quad (3.12)$$

NB: If \mathcal{A} is not unital, we can adjoin an identity as in definition 2.2 for the purpose of the proof and use (2.8), i. e. $\|(0, A)\|_{\bar{\mathcal{A}}} = \|A\|_{\mathcal{A}}$ where needed. Obviously $\|E_\alpha\| \leq 1$, but also

$$\begin{aligned} \forall A_i \in \alpha : (E_\alpha A_i - A_i)(E_\alpha A_i - A_i)^* &= (E_\alpha - \mathbf{1}) A_i A_i^* (E_\alpha - \mathbf{1}) = (\mathbf{1} - E_\alpha) A_i A_i^* (\mathbf{1} - E_\alpha) \\ &\leq \sum_{i=1}^{|\alpha|} (\mathbf{1} - E_\alpha) A_i A_i^* (\mathbf{1} - E_\alpha) = \frac{1}{\mathbf{1} + |\alpha| F_\alpha} F_\alpha \frac{1}{\mathbf{1} + |\alpha| F_\alpha} \\ &= \sqrt{F_\alpha} \left(\frac{1}{\mathbf{1} + |\alpha| F_\alpha} \right)^2 \sqrt{F_\alpha} \stackrel{(2.123c)}{\leq} \sqrt{F_\alpha} \frac{1}{\mathbf{1} + |\alpha| F_\alpha} \sqrt{F_\alpha} \\ &= \frac{1}{|\alpha|} \left(\mathbf{1} - \frac{1}{\mathbf{1} + |\alpha| F_\alpha} \right) \leq \frac{1}{|\alpha|} \mathbf{1}, \quad (3.13) \end{aligned}$$

¹Such a family $\{E_\alpha\}_{\alpha \in U}$ is called a *net*.

²A directed set is a nonempty set with a preorder \leq such that every pair of elements $a, b \in U$ has an *upper bound* $c \in U$: $a \leq c \wedge b \leq c$.

i. e.

$$\|E_\alpha A_i - A_i\|^2 \leq \frac{1}{|\alpha|} \quad (3.14)$$

and since every $A \in \mathcal{I}$ belongs to some family

$$\forall A \in \mathcal{I} : \|E_\alpha A - A\| \rightarrow 0. \quad (3.15)$$

Finally

$$E_\alpha - E_\beta = \frac{1}{\mathbf{1} + |\beta|F_\beta} - \frac{1}{\mathbf{1} + |\alpha|F_\alpha} \quad (3.16)$$

and $\alpha \geq \beta$ implies $|\alpha|F_\alpha \geq |\beta|F_\beta$ and $E_\alpha \geq E_\beta$. \square

Theorem 3.8. *Every closed two sided ideal \mathcal{I} of a C^* -algebra \mathcal{A} is self-adjoint and the factor algebra \mathcal{A}/\mathcal{I} with the norm*

$$\|\hat{A}\|_{\mathcal{A}/\mathcal{I}} = \inf_{I \in \mathcal{I}} \|A + I\|_{\mathcal{A}} \quad (3.17)$$

is a C^* -algebra.

Proof. If $\{E_\alpha\}_{\alpha \in U}$ is an approximate identity of \mathcal{I} , then

$$\forall A \in \mathcal{I} : \|A^* E_\alpha - A^*\|_{\mathcal{A}} = \|E_\alpha A - A\|_{\mathcal{A}} \rightarrow 0. \quad (3.18)$$

Since $A^* E_\alpha \in \mathcal{I}$ and \mathcal{I} is closed, we have $A^* \in \mathcal{I}$. Again, if \mathcal{A} is not unital, we can adjoin an identity as in definition 2.2 to be able to write

$$A - E_\alpha A + I - E_\alpha I = (\mathbf{1} - E_\alpha)(A + I) \quad (3.19)$$

and find

$$\begin{aligned} \limsup_{\alpha} \|A - E_\alpha A\|_{\mathcal{A}} &= \limsup_{\alpha} \|A - E_\alpha A\|_{\bar{\mathcal{A}}} \\ &= \limsup_{\alpha} \|A - E_\alpha A + I - E_\alpha I\|_{\bar{\mathcal{A}}} = \limsup_{\alpha} \|(\mathbf{1} - E_\alpha)(A + I)\|_{\bar{\mathcal{A}}} \\ &\leq \limsup_{\alpha} \|\mathbf{1} - E_\alpha\|_{\bar{\mathcal{A}}} \|A + I\|_{\bar{\mathcal{A}}} \leq \|A + I\|_{\bar{\mathcal{A}}} = \|A + I\|_{\mathcal{A}}, \end{aligned} \quad (3.20)$$

because $0 \leq E_\alpha \leq \mathbf{1}$, or $\sigma(\mathbf{1} - E_\alpha) \subseteq [0, 1]$. Therefore

$$\begin{aligned} \|\hat{A}\|_{\mathcal{A}/\mathcal{I}} &= \inf_{I \in \mathcal{I}} \|A + I\|_{\mathcal{A}} \geq \limsup_{\alpha} \|A - E_\alpha A\|_{\mathcal{A}} \\ &\geq \liminf_{\alpha} \|A - \underbrace{E_\alpha A}_{\in \mathcal{I}}\|_{\mathcal{A}} \geq \inf_{I \in \mathcal{I}} \|A + I\|_{\mathcal{A}} = \|\hat{A}\|_{\mathcal{A}/\mathcal{I}}, \end{aligned} \quad (3.21)$$

i. e.

$$\|\hat{A}\|_{\mathcal{A}/\mathcal{I}} = \lim_{\alpha} \|A - E_{\alpha}A\|_{\mathcal{A}} \quad (3.22)$$

and we get the C^* -property on \mathcal{A}/\mathcal{I} from the C^* -property on \mathcal{A}

$$\begin{aligned} \|\hat{A}\|_{\mathcal{A}/\mathcal{I}}^2 &= \lim_{\alpha} \|A - E_{\alpha}A\|_{\mathcal{A}}^2 = \lim_{\alpha} \|A - E_{\alpha}A\|_{\bar{\mathcal{A}}}^2 \\ &= \lim_{\alpha} \|(A - E_{\alpha}A)(A - E_{\alpha}A)^*\|_{\bar{\mathcal{A}}} = \lim_{\alpha} \|(\mathbf{1} - E_{\alpha})AA^*(\mathbf{1} - E_{\alpha})\|_{\bar{\mathcal{A}}} \\ &= \lim_{\alpha} \|(\mathbf{1} - E_{\alpha})AA^*(\mathbf{1} - E_{\alpha}) + (\mathbf{1} - E_{\alpha})I(\mathbf{1} - E_{\alpha})\|_{\bar{\mathcal{A}}} \\ &= \lim_{\alpha} \|(\mathbf{1} - E_{\alpha})(AA^* + I)(\mathbf{1} - E_{\alpha})\|_{\bar{\mathcal{A}}} \\ &\leq \|AA^* + I\|_{\bar{\mathcal{A}}} = \|AA^* + I\|_{\mathcal{A}} \end{aligned} \quad (3.23)$$

for arbitrary $I \in \mathcal{I}$. Thus

$$\|\hat{A}\|_{\mathcal{A}/\mathcal{I}}^2 \leq \inf_{I \in \mathcal{I}} \|AA^* + I\|_{\mathcal{A}} = \|\hat{A}\hat{A}^*\|_{\mathcal{A}/\mathcal{I}} \leq \|\hat{A}\|_{\mathcal{A}/\mathcal{I}}\|\hat{A}^*\|_{\mathcal{A}/\mathcal{I}} \quad (3.24)$$

i. e.

$$\|\hat{A}\|_{\mathcal{A}/\mathcal{I}} \leq \|\hat{A}^*\|_{\mathcal{A}/\mathcal{I}} \quad (3.25)$$

and

$$\|\hat{A}^*\|_{\mathcal{A}/\mathcal{I}} \leq \|\hat{A}\|_{\mathcal{A}/\mathcal{I}} \quad (3.26)$$

from replacing \hat{A} by \hat{A}^* in the above argument. This implies the C^* -property

$$\|\hat{A}\|_{\mathcal{A}/\mathcal{I}}^2 = \|\hat{A}\hat{A}^*\|_{\mathcal{A}/\mathcal{I}}. \quad (3.27)$$

□

3.3 Continuity

Using this theorem, we can prove a non-obvious result, with profound consequences:

Theorem 3.9. *Let \mathcal{A} be a unital Banach $*$ -algebra and \mathcal{B} a C^* -algebra and $\pi : \mathcal{A} \rightarrow \mathcal{B}$ a $*$ -morphism. Then π is continuous and even*

$$\forall A \in \mathcal{A} : \|\pi(A)\| \leq \|A\|. \quad (3.28)$$

In addition, if \mathcal{A} is a C^ -algebra, then $\mathcal{B}_{\pi} = \pi(\mathcal{A}) \subseteq \mathcal{B}$ is a C^* -subalgebra of \mathcal{B} .*

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Proof. For self-adjoint A , we know that

$$\|\pi(A)\| = \rho(\pi(A)) = \sup_{\lambda \in \sigma(\pi(A))} |\lambda|. \quad (3.29)$$

Now $P = \pi(\mathbf{1}_{\mathcal{A}})$ is a projection

$$P^2 = \pi(\mathbf{1}_{\mathcal{A}})\pi(\mathbf{1}_{\mathcal{A}}) = \pi(\mathbf{1}_{\mathcal{A}}^2) = \pi(\mathbf{1}_{\mathcal{A}}) = P. \quad (3.30)$$

and can be used to define the C^* -algebra (\rightarrow exercise)

$$\mathcal{B}' = P\mathcal{B}P = \{PBP : B \in \mathcal{B}\} \subseteq \mathcal{B} \quad (3.31)$$

with identity $\mathbf{1}_{\mathcal{B}'} = P$. Since

$$\pi(A) = \pi(\mathbf{1}_{\mathcal{A}}A\mathbf{1}_{\mathcal{A}}) = \pi(\mathbf{1}_{\mathcal{A}})\pi(A)\pi(\mathbf{1}_{\mathcal{A}}) = P\pi(A)P \in \mathcal{B}', \quad (3.32)$$

we know that $\pi(\mathcal{A}) \subseteq \mathcal{B}'$ and we can compare the respective spectra: if $(\lambda\mathbf{1}_{\mathcal{A}} - A)^{-1} \in \mathcal{A}$, then $(\lambda P - \pi(A))^{-1} = \pi((\lambda\mathbf{1}_{\mathcal{A}} - A)^{-1}) \in \mathcal{B}'$, i. e. $r_{\mathcal{B}'}(\pi(A)) \supseteq r_{\mathcal{A}}(A)$ or

$$\sigma_{\mathcal{B}'}(\pi(A)) \subseteq \sigma_{\mathcal{A}}(A). \quad (3.33)$$

Note that we have indicated the respective algebras, because \mathcal{B}' and \mathcal{A} are not subsets of each other and theorem 2.21 does *not* apply³. Thus

$$\|\pi(A)\| = \sup_{\lambda \in \sigma(\pi(A))} |\lambda| \leq \sup_{\lambda \in \sigma(A)} |\lambda| \leq \|A\|. \quad (3.34)$$

and for non self-adjoint A we can use

$$\|\pi(A)\|^2 = \|\pi(A^*A)\| \leq \|A^*A\| \leq \|A\|^2 \quad (3.35)$$

to prove $\|\pi(A)\| \leq \|A\|$.

Having shown that π is continuous, we infer that its kernel $\ker \pi$ is a closed two-sided ideal. By theorem 3.8, $\mathcal{A}_{\pi} = \mathcal{A}/\ker \pi$ is then a C^* -algebra.

³In the lecture of November 16, 2018, I wrote

$$\sigma(\pi(A)) = \sigma_{\mathcal{B}}(\pi(A)) = \sigma_{\mathcal{B}'}(\pi(A)) \subseteq \sigma_{\mathcal{A}}(A) = \sigma(A),$$

claiming that an application of theorem 2.21 guarantees that $\sigma_{\mathcal{B}}(\pi(A)) = \sigma_{\mathcal{B}'}(\pi(A))$, since $\mathcal{B}' \subseteq \mathcal{B}$. This is incorrect, as pointed out to me two days later by an observant student, because in general $\mathbf{1}_{\mathcal{B}} \neq \mathbf{1}_{\mathcal{B}'} = P$, in contrast to the requirements of theorem 2.21. Fortunately, it is also unnecessary, because we only need (3.33) here.

Therefore $\hat{\pi} : \mathcal{A}_\pi \rightarrow \mathcal{B}_\pi = \pi(\mathcal{A})$ with $\hat{\pi}(\hat{A}) = \pi(A \in \hat{A})$ is an isomorphism and $\hat{\pi}^{-1} : \mathcal{B}_\pi \rightarrow \mathcal{A}_\pi$ with

$$\hat{\pi}^{-1}(\hat{\pi}(\hat{A})) = \hat{A} \quad (3.36)$$

is well defined. Then from (3.28) applied to $\hat{\pi}^{-1}$ and $\hat{\pi}$, we have

$$\|\hat{A}\| = \|\hat{\pi}^{-1}(\hat{\pi}(\hat{A}))\| \leq \|\hat{\pi}(\hat{A})\| \leq \|\hat{A}\| \quad (3.37)$$

and find that the norms agree

$$\|\hat{A}\|_{\mathcal{A}/\ker \pi} = \|\hat{\pi}(\hat{A})\|_{\mathcal{B}} = \|\pi(A \in \hat{A})\|_{\mathcal{B}}. \quad (3.38)$$

Therefore, whenever a series $\{\pi(A_n)\}_{n \in \mathbf{N}}$ converges to a $B \in \mathcal{B}$, the series $\{\hat{A}_n\}_{n \in \mathbf{N}}$ must converge to a $\hat{A} \in \mathcal{A}_\pi$, because \mathcal{A}_π is complete. By continuity of π and $\hat{\pi}$, $B = \hat{\pi}(\hat{A}) = \pi(A \in \hat{A})$ and $B \in \mathcal{B}_\pi = \pi(\mathcal{A})$, which is therefore closed. \square

Definition 3.10 (isomorphism of C^* -algebras). A C^* -isomorphism $\pi : \mathcal{A} \rightarrow \mathcal{B}$ is a C^* -homomorphism that is *one-to-one* and *onto*, i. e. $\pi(\mathcal{A}) = \mathcal{B}$ and the inverse map $\pi^{-1} : \mathcal{B} \rightarrow \mathcal{A}$ exists.

From linear algebra, we know

Remark 3.11. A C^* -homomorphism $\pi : \mathcal{A} \rightarrow \mathcal{B}$ is a C^* -isomorphism, **iff** its kernel is trivial $\ker \pi = \{0\}$ and $\pi(\mathcal{A}) = \mathcal{B}$.

3.4 Representations

Theorem 1.13 stated that every C^* -algebra is isomorphic to an algebra of operators on a suitable Hilbert space. Thus C^* -homomorphisms that map into the C^* -algebra $\mathcal{L}(\mathcal{H})$ of bounded linear operators on a Hilbert space \mathcal{H} are particularly interesting:

Definition 3.12 (representation of a C^* -algebra). A *representation* of a C^* -algebra \mathcal{A} is a pair (\mathcal{H}, π) consisting of a complex Hilbert space \mathcal{H} and a C^* -homomorphism $\pi : \mathcal{A} \rightarrow \mathcal{L}(\mathcal{H})$. The representation (\mathcal{H}, π) is called *faithful* **iff** $\pi : \mathcal{A} \rightarrow \pi(\mathcal{A})$ is a C^* -isomorphism.

Theorem 3.13. *Let (\mathcal{H}, π) be a representation of the C^* -algebra \mathcal{A} . It is faithful, **iff** it satisfies the following equivalent conditions*

$$\ker \pi = \{0\} \quad (3.39a)$$

$$\forall A \in \mathcal{A} : \|\pi(A)\| = \|A\| \quad (3.39b)$$

$$\forall A > 0 : \pi(A) > 0. \quad (3.39c)$$

Proof. We already know (3.39a) from linear algebra. OTOH, if (3.39a) holds, there is a well defined map $\pi^{-1} : \pi(\mathcal{A}) \rightarrow \mathcal{A}$ with $\pi^{-1}(\pi(A)) = A$ for all A . Using theorem 3.9 twice, i. e.

$$\|A\| = \|\pi^{-1}(\pi(A))\| \leq \|\pi(A)\| \leq \|A\|, \quad (3.40)$$

we obtain (3.39b). Since $A > 0$ implies $\|A\| > 0$, this yields

$$A > 0 \Rightarrow 0 < \|A\| = \|\pi(A)\| \Rightarrow \pi(A) \neq 0. \quad (3.41)$$

Using remark 3.2, we obtain $\pi(A) \geq 0$, i. e. (3.39c). Finally, if $\ker \pi \neq \{0\}$, then $\exists B \in \ker \pi \subseteq \mathcal{A} : B \neq 0 \wedge \pi(B^*B) = \pi(B^*)\pi(B) = 0$. OTOH, $0 < \|B\|^2 = \|B^*B\|$, i. e. $B^*B > 0$ and (3.39c) is false. \square

Definition 3.14 (automorphism of a C^* -algebra). A C^* -isomorphism $\pi : \mathcal{A} \rightarrow \mathcal{A}$ is called a C^* -automorphism.

Corollary 3.15. *If π is a C^* -isomorphism of \mathcal{A} , it is norm preserving*

$$\forall A \in \mathcal{A} : \|\pi(A)\| = \|A\|. \quad (3.42)$$

Proof. This is a direct consequence of theorem 3.13, in particular the equivalence of (3.39a) and (3.39b). \square

Definition 3.16. A subspace $\mathcal{H}_1 \subseteq \mathcal{H}$ is called an *invariant subspace*, also known as (a.k.a.) a *stable subspace*, of a representation (\mathcal{H}, π) of a C^* -algebra, iff

$$\forall A \in \mathcal{A} : \pi(A)\mathcal{H}_1 \subseteq \mathcal{H}_1. \quad (3.43)$$

If \mathcal{H}_1 is a closed subspace of \mathcal{H} and $P_1 = P_1^*$ is the corresponding *orthogonal* projector with $P_1\mathcal{H} = \mathcal{H}_1$, then

$$\forall A \in \mathcal{A} : P_1\pi(A)P_1 = \pi(A)P_1. \quad (3.44)$$

As a consequence

$$\begin{aligned} \pi(A)P_1 &= P_1\pi(A)P_1 = (P_1^*\pi(A)^*P_1^*)^* = (P_1\pi(A)^*P_1)^* \\ &= (\pi(A)^*P_1)^* = P_1^*\pi(A) = P_1\pi(A), \end{aligned} \quad (3.45)$$

i. e.

$$\pi(A)P_1 = P_1\pi(A). \quad (3.46)$$

And indeed this is the necessary *and* sufficient condition for \mathcal{H}_1 to be stable under π .

Remark 3.17. If \mathcal{H}_1 is stable under π and P_1 is the corresponding orthogonal projection, then (\mathcal{H}_1, π_1) with

$$\pi_1(A) = P_1\pi(A)P_1 \quad (3.47)$$

is also a representation. It is called a *subrepresentation*.

Proof.

$$\begin{aligned} \pi_1(A)\pi_1(B) &= P_1\pi(A)P_1P_1\pi(B)P_1 \\ &= P_1\pi(A)\pi(B)P_1 = P_1\pi(AB)P_1 = \pi_1(AB). \end{aligned} \quad (3.48)$$

□

If P_1 is an orthogonal projection, then $\mathbf{1} - P_1 = P_2$ is one as well and projects on the orthogonal subspace \mathcal{H}_2 . Since

$$P_2\pi(A) = (\mathbf{1} - P_1)\pi(A) = \pi(A)(\mathbf{1} - P_1) = \pi(A)P_2, \quad (3.49)$$

\mathcal{H}_2 is also stable under π and (\mathcal{H}_2, π_2) with

$$\pi_2(A) = P_2\pi(A)P_2 \quad (3.50)$$

is again a subrepresentation. In this way, we obtain a *decomposition* of a representation into a direct sum

$$(\mathcal{H}, \pi) = (\mathcal{H}_1, \pi_1) \oplus (\mathcal{H}_2, \pi_2) = (\mathcal{H}_1 \oplus \mathcal{H}_2, \pi_1 \oplus \pi_2) \quad (3.51)$$

where

$$\forall A \in \mathcal{A}, \psi_{1,2} \in \mathcal{H}_{1,2} : \pi(A)(\psi_1 \oplus \psi_2) = \pi_1(A)\psi_1 \oplus \pi_2(A)\psi_2. \quad (3.52)$$

As always, the definition of a representation allows for *trivial representations* with

$$\pi : \mathcal{A} \rightarrow \{0\} \subseteq \mathcal{L}(\mathcal{H}) \quad (3.53)$$

that are not particularly interesting. Moreover, there are nontrivial representations with parts that are not interesting either. The set

$$\mathcal{H}_0 = \bigcap_{A \in \mathcal{A}} \ker(\pi(A)) = \{\psi \in \mathcal{H} : \pi(A)\psi = 0, \forall A \in \mathcal{A}\} \quad (3.54)$$

is obviously a linear subspace and invariant under $\pi(\mathcal{A})$. The corresponding subrepresentation (\mathcal{H}_0, π_0) has

$$\pi_0(A) = P_0\pi(A)P_0 = 0. \quad (3.55)$$

Representation with a trivial subrepresentation are called degenerate. **OTOH**, the interesting cases are covered by

Definition 3.18 (*nondegenerate representation*). A representation (\mathcal{H}, π) is called nondegenerate, **iff** the subspace annihilated by $\pi(\mathcal{A})$ is trivial: $\mathcal{H}_0 = \{0\}$.

Particularly interesting is the case when there is

Definition 3.19 (*cyclic vector*). A vector $\Omega \in \mathcal{H}$ is called cyclic in \mathcal{H} for a set $\mathcal{B} \subseteq \mathcal{L}(\mathcal{H})$, if the linear span of $\{B\Omega : B \in \mathcal{B}\}$ is dense in \mathcal{H} .

Then we have

Definition 3.20. A *cyclic representation* of a C^* -algebra \mathcal{A} is a triple $(\mathcal{H}, \pi, \Omega)$ with (\mathcal{H}, π) a representation of \mathcal{A} and Ω a cyclic vector for $\pi(\mathcal{A})$ in \mathcal{H} .

It is obvious that every cyclic representation is nondegenerate. Indeed, if this were not the case, then

$$\exists B \in \mathcal{A} : \pi(B)\Omega \neq 0 \wedge \forall A \in \mathcal{A} : 0 = \pi(A)\pi(B)\Omega = \pi(AB)\Omega, \quad (3.56)$$

which implies $\pi(B)\Omega = 0$ for $A = \mathbf{1}$ in a unital algebra or A an approximate identity, which exists by theorem 3.7.

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It is less obvious that

Theorem 3.21. *If (\mathcal{H}, π) is a nondegenerate representation of a C^* -algebra, it is a direct sum of a family of cyclic subrepresentations.*

Before we can prove the theorem, we need a precise

Definition 3.22 (*direct sum of representations*). Let $\{(\mathcal{H}_i, \pi_i)\}_{i \in I}$ be a family⁴ of representations of a C^* -algebra \mathcal{A} . Then the direct sum of representation spaces

$$\mathcal{H} = \bigoplus_{i \in I} \mathcal{H}_i \quad (3.57)$$

is well defined⁵ and the action of the direct sum of representations

$$\pi = \bigoplus_{i \in I} \pi_i \quad (3.58)$$

⁴The index set I is not necessarily countable.

⁵As above in the proof of theorem 3.7, we can order the finite subsets of I by inclusion to build a directed set. The elements of \mathcal{H} are then families $\{\psi_i\}_{i \in \alpha}$ such that

$$\lim_{|\alpha| \rightarrow \infty} \sum_{i \in \alpha} \|\psi_i\|_i^2 < \infty.$$

This way we obtain a norm and corresponding inner product and can complete the direct sum with respect to (**wrt**) the topology induced by this norm.

is defined as a bounded operator

$$\pi(A) \left(\bigoplus_{i \in I} \psi_i \right) = \bigoplus_{i \in I} (\pi_i(A) \psi_i) \quad (3.59)$$

since all $\pi_i(A)$ are bounded operators with $\|\pi_i(A)\| \leq \|A\|$.

Proof. Choose⁶ a maximal family $\{\Omega_i\}_{i \in I}$ of non-zero $\Omega_i \in \mathcal{H}$ with

$$\forall A, B \in \mathcal{A}, \forall i \neq j \in I : (\pi(A)\Omega_i, \pi(B)\Omega_j) = 0. \quad (3.60)$$

We can then define

$$\forall i \in I : \mathcal{H}_i = \overline{\pi(\mathcal{A})\Omega_i}, \quad (3.61)$$

which are, by construction, invariant subspaces and we obtain subrepresentations

$$\pi_i(A) = P_i \pi(A) P_i, \quad (3.62)$$

and a family $\{(\mathcal{H}_i, \pi_i, \Omega_i)\}_{i \in I}$ of cyclic representations. Since (\mathcal{H}, π) is non-degenerate and the family $\{\Omega_i\}_{i \in I}$ is maximal, there is no non-zero $\psi \in \mathcal{H}$ with $(\psi, \phi) = 0, \forall \phi \in \bigoplus_{i \in I} \mathcal{H}_i$. \square

This theorem shows that we essentially know *all* representations, when we know all cyclic representations. This is important, because we will soon learn that the cyclic representations are intimately related to the physical states, i.e. the normalized positive linear functionals on the C^* -algebra of (complexified) observables.

Close relatives of the cyclic representations are

Definition 3.23 (*irreducible representation*). A self-adjoint set \mathcal{B} of bounded operators on a Hilbert space \mathcal{H} is called irreducible on \mathcal{H} , if there are no closed subspaces of \mathcal{H} invariant under \mathcal{B} other than the trivial $\{0\}$ and \mathcal{H} itself. Analogously, a representation (\mathcal{H}, π) of a C^* -algebra \mathcal{A} is called irreducible **iff** $\pi(\mathcal{A})$ is irreducible on \mathcal{H} .

Theorem 3.24. *Let \mathcal{B} be a self-adjoint set of bounded operators on a Hilbert space \mathcal{H} . The following are equivalent:*

1. \mathcal{B} is irreducible.
2. only multiples of $\mathbf{1} \in \mathcal{L}(\mathcal{H})$ commute with all $B \in \mathcal{B}$, i.e. the commutant $\mathcal{B}' = \{\lambda \mathbf{1} : \lambda \in \mathbf{C}\}$.

⁶For the existence of such a family, in the case of a not countable I , we have to assume the *axiom of choice* or, equivalently, *Zorn's lemma*.

3. every nonzero $\psi \in \mathcal{H}$ is cyclic for \mathcal{B} in \mathcal{H} , unless $\mathcal{B} = \{0\}$ or $\mathcal{H} = \mathbf{C}$.

Proof. We prove the cyclic dependence:

1. (1)→(3): If there is a nonzero $\psi \in \mathcal{H}$ that is not cyclic, then the orthogonal complement of $\mathcal{B}\psi$ is a non-empty invariant⁷ subspace and \mathcal{B} is reducible on \mathcal{H} (unless $\mathcal{B} = \{0\}$ or $\mathcal{H} = \mathbf{C}$).
2. (3)→(2): If $B \in \mathcal{B}'$, then $B^* \in \mathcal{B}'$, since \mathcal{B} is self-adjoint. Also $B + B^* \in \mathcal{B}'$ and $(B - B^*)/i \in \mathcal{B}'$ and if $\mathcal{B}' \neq \{\lambda \mathbf{1}\}_{\lambda \in \mathbf{C}}$, there are $C = C^* \in \mathcal{B}'$ with $C \neq \lambda \mathbf{1}, \forall \lambda \in \mathbf{C}$. The corresponding spectral projections P of C are then also in \mathcal{B}' and a $\psi \in \mathcal{H}$ with $(1 - P)\psi = 0$ cannot be cyclic.
3. (2)→(1): If \mathcal{B} is not irreducible on \mathcal{H} , then there is a closed subspace $\mathcal{K} \subseteq \mathcal{H}$ that is stable under \mathcal{B} and the corresponding non-trivial projector $P_{\mathcal{K}} \in \mathcal{B}'$.

□

Note that *not* all cyclic representations are irreducible: we have just proven that all non-zero vectors of an irreducible representation are cyclic. But there are cyclic representations where *not all* non-zero vectors are cyclic.

Definition 3.25 (*unitary equivalence*). Two representations (\mathcal{H}_1, π_1) and (\mathcal{H}_2, π_2) are called unitarily equivalent, **iff** there is a unitary operator $U : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ such that

$$\forall A \in \mathcal{A} : \pi_2(A) = U\pi_1(A)U^* . \quad (3.63)$$

i. e. the diagram

$$\begin{array}{ccc} \mathcal{H}_1 & \xrightarrow{\pi_1(A)} & \mathcal{H}_1 \\ U \downarrow & & \downarrow U \\ \mathcal{H}_2 & \xrightarrow{\pi_2(A)} & \mathcal{H}_2 \end{array} \quad (3.64)$$

commutes for all $A \in \mathcal{A}$.

Remark 3.26. Obviously, given a representation (\mathcal{H}_1, π_1) and a unitary operator $U : \mathcal{H}_1 \rightarrow \mathcal{H}_2$, we obtain another representation (\mathcal{H}_2, π_2) from (3.63).

⁷ $(\mathcal{B}\psi, B(\mathcal{B}\psi)^\perp) = (B^*\mathcal{B}\psi, (\mathcal{B}\psi)^\perp) = (\mathcal{B}\psi, (\mathcal{B}\psi)^\perp) = 0$

—4— STATES

As before, the positive linear functionals on a C^* -algebra of observables will play the role of physical states.

Definition 4.1 (*dual space*). The space of continuous linear functionals $\omega : \mathcal{A} \rightarrow \mathbf{C}$ on the C^* -algebra \mathcal{A} is denoted \mathcal{A}^* .

We can define a natural norm on \mathcal{A}^* by

$$\|\omega\| = \sup_{A \in \mathcal{A}, \|A\|=1} |\omega(A)| . \quad (4.1)$$

Definition 4.2 (*positive linear functional, state*). A linear functional $\omega : \mathcal{A} \rightarrow \mathbf{C}$ on the C^* -algebra \mathcal{A} is called positive, **iff**

$$\forall A \in \mathcal{A} : \omega(A^*A) \geq 0 . \quad (4.2)$$

A positive $\omega : \mathcal{A} \rightarrow \mathbf{C}$ with $\|\omega\| = 1$ is called a state.

Typical examples that should be familiar from the **QM** lecture

- *pure states, a.k.a. vector states*:

$$\omega(A) = (\Omega, \pi(A)\Omega) \quad (4.3)$$

with $\Omega \in \mathcal{H}$ normalized, i. e. $(\Omega, \Omega) = 1$. Positivity is obvious

$$\begin{aligned} \omega(A^*A) &= (\Omega, \pi(A^*A)\Omega) = (\Omega, \pi(A^*)\pi(A)\Omega) \\ &= (\Omega, \pi(A)^*\pi(A)\Omega) = (\pi(A)\Omega, \pi(A)\Omega) \geq 0 \end{aligned} \quad (4.4)$$

and the normalization $\|\omega\| = 1$ will be a consequence of theorem 4.4 below.

- statistical mixtures

$$\omega(A) = \sum_i p_i \omega_i(A) \quad (4.5)$$

with $p_i \geq 0$ and $\sum_i p_i = 1$.

- density matrices

$$\omega(A) = \text{tr}(\rho \pi(A)) \quad (4.6)$$

with $\rho = \rho^* > 0$ and $\text{tr} \rho = 1$.

These will be discussed in more detail below.

Lecture 12: Fri, 23. 11. 2018

In the following we will frequently use the **CSI** (1.48b), which we have already derived on page 11 when we discussed algebras of observables, before introducing C^* -algebras. The identical arguments will also establish

Lemma 4.3 (*Cauchy-Schwarz Inequality*). *For every positive linear functional ω on a C^* -algebra \mathcal{A} , we have $\forall A, B \in \mathcal{A}$*

$$\omega(B^*A) = \overline{\omega(A^*B)} \quad (4.7a)$$

$$|\omega(A^*B)| \leq \sqrt{\omega(A^*A)} \sqrt{\omega(B^*B)}. \quad (4.7b)$$

We will now see that every positive $\omega : \mathcal{A} \rightarrow \mathbf{C}$ is continuous. ,

Theorem 4.4 (continuity of positive functionals). *For all $\omega : \mathcal{A} \rightarrow \mathbf{C}$, the following are equivalent*

1. ω is positive
2. ω is continuous and

$$\|\omega\| = \lim_{\alpha} \omega(E_{\alpha}^2) \quad (4.8)$$

for some approximate identity $\{E_{\alpha}\}$ of \mathcal{A} .

Furthermore if ω is positive, then $\forall A, B \in \mathcal{A}$

$$\omega(A^*) = \overline{\omega(A)} \quad (4.9a)$$

$$|\omega(A)|^2 \leq \omega(A^*A) \|\omega\| \quad (4.9b)$$

$$|\omega(A^*BA)| \leq \omega(A^*A) \|B\| \quad (4.9c)$$

$$\|\omega\| = \sup_{A \in \mathcal{A}, \|A\|=1} \omega(A^*A) \quad (4.9d)$$

and

$$\|\omega\| = \lim_{\alpha} \omega(E_{\alpha}) \quad (4.10)$$

for any approximate identity $\{E_{\alpha}\}$ of \mathcal{A} .

Proof. First, assume that ω is positive, i. e. (1), and let $\{A_n\}_{n \in \mathbf{N}}$ be a sequence with $A_n \geq 0$ and $\|A_n\| \leq 1$. Then for *all* sequences $\{\lambda_n\}_{n \in \mathbf{N}}$ with $\lambda_n \geq 0$ and $\sum_{n \in \mathbf{N}} \lambda_n < \infty$

$$\sum_{n=1}^N \lambda_n A_n \xrightarrow{N \rightarrow \infty} A \geq 0 \quad (4.11)$$

converges monotonically from below. Thus, by positivity and linearity of ω

$$\sum_{n=1}^N \lambda_n \omega(A_n) \leq \omega(A) < \infty. \quad (4.12)$$

This can only be true for *all* such sequences $\{\lambda_n\}_{n \in \mathbf{N}}$ *independently* of $\{A_n\}_{n \in \mathbf{N}}$, if the $\{\omega(A_n)\}_{n \in \mathbf{N}}$ are *uniformly* bounded

$$M_+ = \sup_{A \geq 0, \|A\|=1} \omega(A) < \infty. \quad (4.13)$$

We can decompose each $A \in \mathcal{A}$ with $\|A\| = 1$ into two self-adjoint and four positive $\{A_k\}$ with $\|A_k\| \leq 1$

$$A = \sum_{k=1}^4 i^k A_k \quad (4.14)$$

and therefore

$$\|\omega\| = \sup_{\|A\|=1} |\omega(A)| \leq 4M_+. \quad (4.15)$$

Since this entails

$$|\omega(A)| \leq 4M_+ \|A\|, \quad (4.16)$$

we have established that $\omega : \mathcal{A} \rightarrow \mathbf{C}$ is continuous. For any $E_\alpha \in \mathcal{A}$ we can use the **CSI**

$$|\omega(AE_\alpha)|^2 \stackrel{(1.48b)}{\leq} \omega(A^*A)\omega(E_\alpha^*E_\alpha) \leq M_+ \|A\|^2 \omega(E_\alpha^2) \quad (4.17)$$

and if $\{E_\alpha\}$ is an approximate identity, we can take the limit $|\alpha| \rightarrow \infty$

$$|\omega(A)|^2 \leq M_+ M \|A\|^2 \quad (4.18)$$

where

$$M = \sup_{\alpha} \omega(E_\alpha^2). \quad (4.19)$$

Taking the supremum over $\|A\| = 1$, we obtain

$$\|\omega\|^2 \leq M_+ M. \quad (4.20)$$

However, $\|E_\alpha\| \leq 1$ and $E_\alpha \geq 0$ by definition and therefore

$$M = \sup_\alpha \omega(E_\alpha^2) \leq M_+ = \sup_{A \geq 0, \|A\|=1} \omega(A) \leq \sup_{\|A\|=1} \omega(A) = \|\omega\|. \quad (4.21)$$

This entails

$$\|\omega\|^2 \leq M_+ M \leq M_+ \|\omega\| \quad (4.22a)$$

$$\|\omega\|^2 \leq M_+ M \leq M \|\omega\| \quad (4.22b)$$

and the inequalities must be equalities

$$\|\omega\| = M_+ = M = \lim_\alpha \omega(E_\alpha^2), \quad (4.23)$$

i. e. we have established (2). While we're at it, we observe that since $E_\alpha^2 \leq E_\alpha$

$$\|\omega\| = \lim_\alpha \omega(E_\alpha^2) \leq \lim_\alpha \omega(E_\alpha) \leq \|\omega\| \quad (4.24)$$

i. e. (4.10).

Conversely, assume that (2) holds and **WLOG** $\|\omega\| = 1$. If \mathcal{A} is unital, then

$$\begin{aligned} \|\mathbf{1} - E_\alpha^2\| &\leq \|\mathbf{1} - E_\alpha\| + \|E_\alpha - E_\alpha^2\| \\ &\leq \|\mathbf{1} - E_\alpha\| + \|\mathbf{1} - E_\alpha\| \|E_\alpha\| = \|\mathbf{1} - E_\alpha\| (1 + \|E_\alpha\|) \end{aligned} \quad (4.25)$$

therefore

$$\lim_\alpha E_\alpha^2 = \mathbf{1} \quad (4.26)$$

and

$$\omega(\mathbf{1}) = \lim_\alpha \omega(E_\alpha^2) = \|\omega\| = 1 \quad (4.27)$$

from (2). **OTOH**, if \mathcal{A} is not unital, we can adjoin a unit and extend $\omega : \mathcal{A} \rightarrow \mathbf{C}$ to

$$\begin{aligned} \tilde{\omega} : \mathbf{C}\mathbf{1} + \mathcal{A} &\rightarrow \mathbf{C} \\ (\lambda, A) &\mapsto \lambda + \omega(A). \end{aligned} \quad (4.28)$$

Then

$$A - AE_\alpha^2 = (A - AE_\alpha) + (A - AE_\alpha)E_\alpha, \quad (4.29)$$

i. e.

$$\lim_\alpha AE_\alpha^2 = A. \quad (4.30)$$

Furthermore

$$|\omega(A)| \leq \|\omega\| \|A\| \quad (4.31)$$

i. e.

$$|\omega(A)| \leq \|A\| \quad (4.32)$$

and

$$\begin{aligned} |\tilde{\omega}(\lambda \mathbf{1} + A)| &= |\lambda + \omega(A)| = \lim_{\alpha} |\lambda \omega(E_{\alpha}^2) + \omega(AE_{\alpha}^2)| \\ &\stackrel{(4.32)}{\leq} \limsup_{\alpha} \|\lambda E_{\alpha}^2 + AE_{\alpha}^2\|_{\mathcal{A}} \leq \sup_{\|B\|=1} \|\lambda B + AB\|_{\mathcal{A}} = \|\lambda \mathbf{1} + A\|_{\bar{\mathcal{A}}}. \end{aligned} \quad (4.33)$$

thus we have in any case

$$|\omega(\mathbf{1})| = \|\mathbf{1}\| = 1 = \|\omega\|. \quad (4.34)$$

In order to show that $\omega(A)$ is real for self adjoint A , we observe that for

$$\omega(A) = \alpha + i\beta \quad (\alpha, \beta \in \mathbf{R}) \quad (4.35)$$

we have

$$\omega(A + i\gamma \mathbf{1}) = \alpha + i(\beta + \gamma). \quad (4.36)$$

For any self-adjoint A , the spectrum satisfies

$$\sigma(A) + i\gamma \subseteq [-\|A\|, \|A\|] + i\gamma \quad (4.37)$$

and since $A + i\gamma \mathbf{1}$ is normal, we have

$$\|A + i\gamma \mathbf{1}\| = \rho(A + i\gamma \mathbf{1}) = \sqrt{\|A\|^2 + \gamma^2}. \quad (4.38)$$

OTOH

$$|\omega(A + i\gamma \mathbf{1})| = |\alpha + i(\beta + \gamma)| = \sqrt{\alpha^2 + (\beta + \gamma)^2} \geq |\beta + \gamma| \quad (4.39)$$

and therefore

$$\forall \gamma \in \mathbf{R} : |\beta + \gamma| \leq \sqrt{\|A\|^2 + \gamma^2}. \quad (4.40)$$

Choosing $\gamma = \delta \|A\|^2 \beta / |\beta|$, we find

$$\forall \delta \in \mathbf{R} : \beta^2 + 2\delta |\beta| \|A\|^2 \leq \|A\|^2, \quad (4.41)$$

which can hold for $\delta \rightarrow \infty$ only if $\beta = 0$, i. e. $\omega(A)$ must be real if $A = A^*$. However, since $A^*A \geq 0$

$$\left\| \mathbf{1} - \frac{A^*A}{\|A^*A\|} \right\| = \left\| \mathbf{1} - \frac{A^*A}{\|A\|^2} \right\| \leq 1 \quad (4.42)$$

and therefore, using (4.31),

$$\left| \omega(\mathbf{1}) - \frac{\omega(A^*A)}{\|A\|^2} \right| \leq 1. \quad (4.43)$$

With $\omega(\mathbf{1}) = 1$ and $\omega(A^*A) \in \mathbf{R}$ this entails the positivity

$$\omega(A^*A) \geq 0. \quad (4.44)$$

The remaining results are straightforward: (4.9a) follows from (4.7a) with $B = E_\alpha$ in the limit $\alpha \rightarrow \infty$ and (4.9b) from (4.7b) in a similar way. Also (4.9c) follows from using (4.7b), this time in the form

$$|\omega(A^*BA)|^2 \leq \omega(A^*A)\omega(A^*B^*BA) \quad (4.45)$$

and noting that

$$A^*B^*BA \leq A^*\|B\|^2A. \quad (4.46)$$

Finally, the estimate (4.9b) implies (4.9d). \square

It is obvious that the sum of two positive linear functionals is again positive and the foregoing theorem tells us that the norm is additive.

Corollary 4.5. *Given two positive linear functionals $\omega_1, \omega_2 \in \mathcal{A}^*$ on a C^* -algebra \mathcal{A} , their sum is positive and*

$$\|\omega_1 + \omega_2\| = \|\omega_1\| + \|\omega_2\| \quad (4.47)$$

and the states form a convex subset of \mathcal{A}^* .

Proof. Using

$$\begin{aligned} \|\omega_1 + \omega_2\| &= \lim_{\alpha} (\omega_1(E_\alpha^2) + \omega_2(E_\alpha^2)) \\ &= \lim_{\alpha} \omega_1(E_\alpha^2) + \lim_{\alpha} \omega_2(E_\alpha^2) = \|\omega_1\| + \|\omega_2\| \end{aligned} \quad (4.48)$$

for a suitable approximate identity $\{E_\alpha\}$ we obtain the additivity and with a similar argument

$$\|\omega\| = \|\lambda\omega_1 + (1 - \lambda)\omega_2\| = \lambda\|\omega_1\| + (1 - \lambda)\|\omega_2\| = 1 \quad (4.49)$$

the correct normalization of the conical sum. \square

We have seen in theorem 2.1 that given a C^* -algebra \mathcal{A} without identity, we can always adjoin an identity to obtain a unital C^* -algebra $\tilde{\mathcal{A}} = \mathbf{C}\mathbf{1} + \mathcal{A}$. We can extend every $\omega \in \mathcal{A}^*$ to $\tilde{\omega} \in \tilde{\mathcal{A}}^*$

$$\tilde{\omega}(\lambda\mathbf{1} + A) = \lambda\|\omega\| + \omega(A). \quad (4.50)$$

It can be shown easily that $\tilde{\omega}$ inherits the positivity, normalization and additivity from ω .

4.1 Pure and Impure States

Just as we used positivity of algebra elements to define an order on the algebra by $A \leq B \Leftrightarrow B - A \geq 0$, we can order the dual space by

$$\forall \omega_1, \omega_2 \in \mathcal{A}^* : \omega_1 \leq \omega_2 \Leftrightarrow \omega_2 - \omega_1 \geq 0. \quad (4.51)$$

We say that ω_2 *majorizes* ω_1 , **iff** $\omega_2 \geq \omega_1$. If a state ω is a *conical sum* of the states ω_1 and ω_2 , i. e.

$$\omega = \lambda \omega_1 + (1 - \lambda) \omega_2 \quad (4.52)$$

with $0 < \lambda < 1$, then

$$\omega - \lambda \omega_1 = (1 - \lambda) \omega_2 \geq 0 \quad (4.53a)$$

$$\omega - (1 - \lambda) \omega_2 = \lambda \omega_1 \geq 0 \quad (4.53b)$$

i. e. ω majorizes both $\lambda \omega_1$ and $(1 - \lambda) \omega_2$. The normalization of the states means that there must be *smallest* states **wrt** to the ordering just defined. Therefore one cannot write all of them as conical sums of majorized states. Thus one can distinguish states ω that can be written as a conical sum of two other states from those that can not.

Definition 4.6 (*pure states*). A state $\omega \in \mathcal{A}^*$ over a C^* -algebra \mathcal{A} is called *pure*, if all $0 \leq \omega' \in \mathcal{A}^*$ majorized by ω are of the form $\omega' = \lambda \omega$ with $0 \leq \lambda \leq 1$.

We will denote the convex cone of all states by $\Omega_{\mathcal{A}} \subset \mathcal{A}^*$ and the subset of all pure states by $\Pi_{\mathcal{A}} \subset \Omega_{\mathcal{A}}$.

4.2 The GNS Construction

Lecture 13: Wed, 28.11.2018

We have seen that for any representation (\mathcal{H}, π) of a C^* -algebra \mathcal{A} and for all $\Omega \in \mathcal{H}$ with $\|\Omega\| = 1$, we obtain a *vector state* $\omega_{\Omega} : A \mapsto (\Omega, \pi(A)\Omega)$. In this section, we will show *by construction* that *every* state ω on \mathcal{A} is a vector state for a suitable cyclic representation $(\mathcal{H}_{\omega}, \pi_{\omega}, \Omega_{\omega})$ and that this representation will be irreducible, **iff** ω is pure.

Since every C^* -algebra \mathcal{A} is, by definition, a Banach space, we can recycle its vector space structure to construct a Hilbert space, **iff** we can find a suitable nondegenerate *sesquilinear form*

$$\begin{aligned} \langle \cdot, \cdot \rangle : \mathcal{A} \times \mathcal{A} &\rightarrow \mathbf{C} \\ (A, B) &\mapsto \langle A, B \rangle \end{aligned} \quad (4.54)$$

with $\forall A, B, C \in \mathcal{A}$ and $\forall \beta, \gamma \in \mathbf{C}$

$$\langle A, B \rangle = \overline{\langle B, A \rangle} \quad (4.55a)$$

$$\langle A, \beta B + \gamma C \rangle = \beta \langle A, B \rangle + \gamma \langle A, C \rangle \quad (4.55b)$$

$$\langle A, A \rangle \geq 0 \quad (4.55c)$$

$$\langle A, A \rangle = 0 \Leftrightarrow A = 0. \quad (4.55d)$$

In fact, we can construct a sesquilinear form from any positive linear functional ω on an algebra

$$\langle A, B \rangle_\omega = \omega(A^* B). \quad (4.56)$$

Indeed, all properties (4.55) except (4.55d) can be verified trivially, where (4.55a) follows from the positivity of ω via (4.7a). However, there will in general be $0 \neq A \in \mathcal{A}$ with $\omega(A^* A) = 0$.

Fortunately, these elements form a left ideal

$$\mathcal{I}_\omega = \{A \in \mathcal{A} : \omega(A^* A) = 0\}, \quad (4.57)$$

because

$$\begin{aligned} \forall I \in \mathcal{I}_\omega, A \in \mathcal{A} : 0 &\leq \omega((AI)^* AI) = \omega(I^* A^* AI) \\ &\leq \omega(I^* I) \|A^* A\| = \omega(I^* I) \|A\|^2 = 0, \end{aligned} \quad (4.58)$$

by result (4.9c) of theorem 4.4. Thus we have shown $\forall I \in \mathcal{I}_\omega, A \in \mathcal{A} : AI \in \mathcal{I}_\omega$. Having identified this left ideal, we can form the factor space $\mathcal{H}_\omega^0 = \mathcal{A}/\mathcal{I}_\omega$ consisting of the equivalence classes

$$\psi_\omega^A = \{\hat{A} : \hat{A} = A + I, I \in \mathcal{I}_\omega\} \in \mathcal{H}_\omega^0. \quad (4.59)$$

By construction, \mathcal{H}_ω^0 is a linear space and we get again a sesquilinear form

$$\begin{aligned} \langle \cdot, \cdot \rangle : \mathcal{H}_\omega^0 \times \mathcal{H}_\omega^0 &\rightarrow \mathbf{C} \\ (\psi_\omega^A, \psi_\omega^B) &\mapsto \langle \psi_\omega^A, \psi_\omega^B \rangle = \omega(A^* B), \end{aligned} \quad (4.60)$$

since we can use the CSI to show that it does not depend on which representative of each equivalence class is used

$$\forall I \in \mathcal{I}_\omega, A \in \mathcal{A} : |\omega(A^* I)| \leq \sqrt{\omega(A^* A)} \sqrt{\omega(I^* I)} = 0, \quad (4.61)$$

i. e.

$$\forall I \in \mathcal{I}_\omega, A \in \mathcal{A} : \omega(A^*(B + I)) = \omega(A^* B) + \omega(A^* I) = \omega(A^* B). \quad (4.62)$$

Conversely

$$\forall I \in \mathcal{I}_\omega, \mathcal{A} \ni A \notin \mathcal{I}_\omega : \omega((A + I)^*(A + I)) = \omega(A^*A) > 0 \quad (4.63)$$

by the definition of \mathcal{I}_ω . Thus $\langle \cdot, \cdot \rangle$ turns \mathcal{H}_ω^0 into a proper pre-Hilbert space that can be completed to a Hilbert space $\mathcal{H}_\omega = \overline{\mathcal{H}_\omega^0}$ and $\langle \cdot, \cdot \rangle$ can be extended to (\cdot, \cdot) on \mathcal{H}_ω by the continuity of ω .

Having constructed a Hilbert space is not yet enough for a representation, we still need to find a homomorphism $\pi_\omega : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H}_\omega)$. Fortunately, the algebra structure of \mathcal{A} suffices to allow a *canonical* construction

$$\begin{aligned} \pi_\omega(A) : \mathcal{H}_\omega &\rightarrow \mathcal{H}_\omega \\ \psi_\omega^B &\mapsto \pi_\omega(A)\psi_\omega^B = \psi_\omega^{AB}. \end{aligned} \quad (4.64)$$

This definition has *all* the required properties:

- it is well defined, because it's independent of the representative

$$\forall I \in \mathcal{I}_\omega : A(B + I) = AB + \underbrace{AI}_{\in \mathcal{I}_\omega} \quad (4.65)$$

- it's linear due to the linear structure of the factor space

$$\begin{aligned} \pi_\omega(\alpha_1 A_1 + \alpha_2 A_2)\psi_\omega^B &= \psi_\omega^{(\alpha_1 A_1 + \alpha_2 A_2)B} = \alpha_1 \psi_\omega^{A_1 B} + \alpha_2 \psi_\omega^{A_2 B} \\ &= \alpha_1 \pi_\omega(A_1)\psi_\omega^B + \alpha_2 \pi_\omega(A_2)\psi_\omega^B \end{aligned} \quad (4.66)$$

- it's an algebra homomorphism due to the associativity of the algebra

$$\begin{aligned} \pi_\omega(A_1)\pi_\omega(A_2)\psi_\omega^B &= \pi_\omega(A_1)\psi_\omega^{A_2 B} \\ &= \psi_\omega^{A_1(A_2 B)} = \psi_\omega^{(A_1 A_2)B} = \pi_\omega(A_1 A_2)\psi_\omega^B \end{aligned} \quad (4.67)$$

- $\pi_\omega(A)$ is bounded

$$\begin{aligned} \|\pi_\omega(A)\psi_\omega^B\|^2 &= \|\psi_\omega^{AB}\|^2 = (\psi_\omega^{AB}, \psi_\omega^{AB}) = \omega((AB)^*AB) \\ &= \omega(B^*A^*AB) \leq \|A^*A\|\omega(B^*B) = \|A\|^2\omega(B^*B) \\ &= \|A\|^2\|\psi_\omega^B\|^2 \end{aligned} \quad (4.68)$$

i. e.

$$\|\pi_\omega(A)\| \leq \|A\|. \quad (4.69)$$

If \mathcal{A} is unital, the cyclic vector Ω_ω is given simply by

$$\Omega_\omega = \psi_\omega^{\mathbf{1}} \quad (4.70)$$

recovering the vector state

$$(\Omega_\omega, \pi_\omega(A)\Omega_\omega) = (\psi_\omega^{\mathbf{1}}, \psi_\omega^A) = \omega(\mathbf{1}^*A) = \omega(A), \quad (4.71)$$

since

$$\pi_\omega(A)\Omega_\omega = \pi_\omega(A)\psi_\omega^{\mathbf{1}} = \psi_\omega^A. \quad (4.72)$$

We verify that

$$\{\pi_\omega(A)\Omega_\omega : A \in \mathcal{A}\} = \{\psi_\omega^A : A \in \mathcal{A}\} = \mathcal{H}_\omega \quad (4.73)$$

i. e. that Ω_ω is cyclic for $(\mathcal{H}_\omega, \pi_\omega)$. **OTOH**, if \mathcal{A} is not unital, we can again adjoin an identity $\mathbf{1}$ to get $\tilde{\mathcal{A}}$ and proceed as above¹. This time, however, while it is again obvious that $\Omega_\omega = \psi_\omega^{\mathbf{1}}$ is cyclic for $\pi_\omega(\tilde{\mathcal{A}})$, it we must also show that it is cyclic for $\pi_\omega(\mathcal{A})$. Since only $\pi_\omega(\mathbf{1})$ is missing from $\pi_\omega(\mathcal{A})$, it suffices to show that $\Omega_\omega = \pi_\omega(\mathbf{1})\Omega_\omega$ itself is already in $\mathcal{H}_\omega = \pi_\omega(\mathcal{A})\Omega_\omega$.

Again, the existence of approximate identities, as ensured by theorem 3.7, comes to the rescue. Let $\{E_\alpha\}$ be an approximate identity for \mathcal{A} , then

$$\begin{aligned} \|\pi_\omega(E_\alpha)\Omega_\omega - \Omega_\omega\|^2 &= \underbrace{\|\Omega_\omega\|^2}_{\omega(\mathbf{1}\mathbf{1})} + \underbrace{\|\pi_\omega(E_\alpha)\Omega_\omega\|^2}_{\omega(E_\alpha E_\alpha)} - 2 \underbrace{(\Omega_\omega, \pi_\omega(E_\alpha)\Omega_\omega)}_{\omega(\mathbf{1}E_\alpha)} \\ &= 1 + \omega(E_\alpha^2) - 2\omega(E_\alpha) \xrightarrow{\alpha} 1 + 1 - 2 = 0 \end{aligned} \quad (4.75)$$

i. e.

$$\lim_\alpha \|\pi_\omega(E_\alpha)\Omega_\omega - \Omega_\omega\| = 0 \quad (4.76)$$

and Ω_ω lies in the closure of $\bigcup_\alpha \pi_\omega(E_\alpha)\Omega_\omega \subset \pi_\omega(\mathcal{A})\Omega_\omega$.

With this construction, we have already almost proved the

¹Note that one can not circumvent $\tilde{\mathcal{A}}$ by directly using a right approximate identity to construct Ω_ω as $\lim_\alpha \psi_\omega^{E_\alpha}$, because, in general, only the limit $\lim_\alpha E_\alpha A = A \in \mathcal{A}$ exists and not the limit $\lim_\alpha E_\alpha \in \mathcal{A}$. Nevertheless, the limit $\lim_\alpha \psi_\omega^{E_\alpha} \in \mathcal{H}_\omega$ exists because \mathcal{H}_ω is complete and then one can argue

$$\pi_\omega(A) \lim_\alpha \psi_\omega^{E_\alpha} = \lim_\alpha \psi_\omega^{AE_\alpha} = \psi_\omega^A \quad (4.74)$$

iff $\{E_\alpha\}_\alpha$ is a left approximate identity. However, we have only proven the existence of right approximate identities above.

Theorem 4.7 (*GNS representation*). *If ω is a state over the C^* -algebra \mathcal{A} , then there exists a cyclic representation $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ of \mathcal{A} with*

$$\forall A \in \mathcal{A} : \omega(A) = (\Omega_\omega, \pi_\omega(A)\Omega_\omega) \quad (4.77)$$

and

$$\|\Omega_\omega\|^2 = (\Omega_\omega, \Omega_\omega) = \omega(\mathbf{1}) = \|\omega\|. \quad (4.78)$$

This representation is unique up to unitary equivalence and called the GNS representation associated to ω .

Proof. After the explicit description of the construction above, it only remains to prove the uniqueness up to unitary equivalence, i. e. that if there is a second cyclic representation $(\mathcal{H}'_\omega, \pi'_\omega, \Omega'_\omega)$ with $\omega(A) = (\Omega'_\omega, \pi'_\omega(A)\Omega'_\omega)$, there is a unitary map $U : \mathcal{H}_\omega \rightarrow \mathcal{H}'_\omega$ such that

$$\begin{array}{ccc} \mathcal{H}_\omega & \xrightarrow{\pi_\omega(A)} & \mathcal{H}_\omega \\ \downarrow U & & \downarrow U \\ \mathcal{H}'_\omega & \xrightarrow{\pi'_\omega(A)} & \mathcal{H}'_\omega \end{array} \quad (4.79a)$$

commutes for all $A \in \mathcal{A}$ and that

$$U\Omega_\omega = \Omega'_\omega. \quad (4.79b)$$

Due to the cyclicity of the representations, we can construct such a U easily by demanding (4.79b) and defining

$$\forall A \in \mathcal{A} : U\psi_\omega^A = U\pi_\omega(A)\Omega_\omega = \pi'_\omega(A)U\Omega_\omega = \pi'_\omega(A)\Omega'_\omega = \psi_\omega'^A \quad (4.80)$$

and extending it to all of \mathcal{H}_ω . This map respects the inner products

$$\begin{aligned} (U\psi_\omega^A, U\psi_\omega^B) &= (U\pi_\omega(A)\Omega_\omega, U\pi_\omega(B)\Omega_\omega) = (\pi'_\omega(A)\Omega'_\omega, \pi'_\omega(B)\Omega'_\omega) \\ &= \omega(A^*B) = (\pi_\omega(A)\Omega_\omega, \pi_\omega(B)\Omega_\omega) = (\psi_\omega^A, \psi_\omega^B) \end{aligned} \quad (4.81)$$

and the existence of the inverse map $U^{-1} : \mathcal{H}'_\omega \rightarrow \mathcal{H}_\omega$ can be verified by explicit construction with $U^{-1}\Omega'_\omega = \Omega_\omega$ and

$$\forall A \in \mathcal{A} : U^{-1}\psi_\omega'^A = U^{-1}\pi'_\omega(A)\Omega'_\omega = \pi_\omega(A)U^{-1}\Omega'_\omega = \pi_\omega(A)\Omega_\omega = \psi_\omega^A. \quad (4.82)$$

□

The unitary equivalence of all representations constructed from a state ω means that the result of the GNS construction is essentially unique. Furthermore, it implies that symmetries, i. e. automorphisms, of a C^* -algebra are realized in the GNS representations by unitary operators.

Corollary 4.8. *If $\phi : \mathcal{A} \rightarrow \mathcal{B}$ is a homomorphism of C^* -algebras, then we have for the GNS representations derived from the states $\omega : \mathcal{B} \rightarrow \mathbf{C}$ and $\omega \circ \phi : \mathcal{A} \rightarrow \mathbf{C}$*

$$\pi_{\omega \circ \phi} = \pi_{\omega} \circ \phi, \quad (4.83)$$

upto unitary equivalence.

Proof. The situation is clarified by the commutative diagram

$$\begin{array}{ccc} & \mathcal{A} & \xrightarrow{\pi_{\omega \circ \phi}} & \mathcal{L}(\mathcal{H}_{\omega \circ \phi}) \\ \omega \circ \phi \swarrow & \downarrow \phi & & \downarrow \cong \\ \mathbf{C} & & & \mathcal{L}(\mathcal{H}_{\omega}) \\ \omega \swarrow & \mathcal{B} & \xrightarrow{\pi_{\omega}} & \end{array} \quad (4.84)$$

Given the representations $(\mathcal{H}_{\omega}, \pi_{\omega}, \Omega_{\omega})$ and $(\mathcal{H}_{\omega \circ \phi}, \pi_{\omega \circ \phi}, \Omega_{\omega \circ \phi})$, we can compute

$$(\Omega_{\omega \circ \phi}, \pi_{\omega \circ \phi}(A)\Omega_{\omega \circ \phi}) = (\omega \circ \phi)(A) = \omega(\phi(A)) = (\Omega_{\omega}, \pi_{\omega}(\phi(A))\Omega_{\omega}) \quad (4.85)$$

and by theorem 4.7 we infer that, up to unitary equivalence,

$$\pi_{\omega \circ \phi} = \pi_{\omega} \circ \phi. \quad (4.86)$$

□

Corollary 4.9 (unitary realization of automorphisms). *If $\tau : \mathcal{A} \rightarrow \mathcal{A}$ is an automorphism of a C^* -algebra that leaves a state $\omega : \mathcal{A} \rightarrow \mathbf{C}$ invariant, i. e. $\omega \circ \tau = \omega$ or*

$$\forall A \in \mathcal{A} : \omega(\tau(A)) = \omega(A), \quad (4.87)$$

then there exists a unique unitary $U_{\omega}^{\tau} : \mathcal{H}_{\omega} \rightarrow \mathcal{H}_{\omega}$ with

$$U_{\omega}^{\tau} \pi_{\omega}(A) (U_{\omega}^{\tau})^{-1} = \pi_{\omega}(\tau(A)) \quad (4.88a)$$

$$U_{\omega}^{\tau} \Omega_{\omega} = \Omega_{\omega}. \quad (4.88b)$$

Proof. This is simple consequence of corollary 4.8 and

$$\omega \circ \tau = \omega. \quad (4.89)$$

□

An important result concerns the relation of the purity of states and irreducibility of the corresponding GNS representations:

Theorem 4.10 (purity vs. irreducibility). *If ω is a state over a C^* -algebra \mathcal{A} and $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ the associated GNS representations of \mathcal{A} , then the following statements are equivalent*

1. the representation $(\mathcal{H}_\omega, \pi_\omega)$ is irreducible
2. ω is pure.

In addition, there is a one-to-one correspondence between the positive functionals $\omega_T \leq \omega$ and positive operators T with $T \leq \mathbf{1}_{\mathcal{H}_\omega}$ and commuting with all operators in $\pi_\omega(\mathcal{A})$. This correspondence is given by

$$\omega_T(A) = (T\Omega_\omega, \pi_\omega(A)\Omega_\omega) . \quad (4.90)$$

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Proof. In order to prove that 2) follows from 1), assume that ω is not pure. Then there is a positive $\rho \neq \lambda\omega, \forall \lambda \in \mathbf{R}_+$ majorized by ω , i. e. $\forall A \in \mathcal{A} : 0 \leq \rho(A^*A) \leq \omega(A^*A)$. Using once more the **CSI**

$$\begin{aligned} |\rho(B^*A)|^2 &\leq \rho(B^*B)\rho(A^*A) \leq \omega(B^*B)\omega(A^*A) \\ &= \|\pi_\omega(B)\Omega_\omega\|^2 \|\pi_\omega(A)\Omega_\omega\|^2 = \|\psi_\omega^B\|^2 \|\psi_\omega^A\|^2, \end{aligned} \quad (4.91)$$

we see that

$$\begin{aligned} \hat{\rho} : \mathcal{H}_\omega \times \mathcal{H}_\omega &\rightarrow \mathbf{C} \\ (\psi_\omega^B, \psi_\omega^A) &\mapsto \rho(B^*A) \end{aligned} \quad (4.92)$$

is a densely defined, bounded sesquilinear functional. By the Riesz representation theorem, there is a bounded operator $T \in \mathcal{L}(\mathcal{H}_\omega)$, such that

$$\rho(B^*A) = (\psi_\omega^B, T\psi_\omega^A) = (\pi_\omega(B)\Omega_\omega, T\pi_\omega(A)\Omega_\omega) . \quad (4.93)$$

Since ρ is not a multiple of ω , this T must not be a multiple of $\mathbf{1}_{\mathcal{H}_\omega}$. From

$$\begin{aligned} 0 &\leq (\pi_\omega(A)\Omega_\omega, T\pi_\omega(A)\Omega_\omega) = \rho(A^*A) \\ &\leq \omega(A^*A) = (\pi_\omega(A)\Omega_\omega, \pi_\omega(A)\Omega_\omega) \end{aligned} \quad (4.94)$$

we conclude

$$0 < T < \mathbf{1}_{\mathcal{H}_\omega} . \quad (4.95)$$

Furthermore

$$\begin{aligned} \forall C \in \mathcal{A} : (\pi_\omega(B)\Omega_\omega, T\pi_\omega(C)\pi_\omega(A)\Omega_\omega) &= \rho(B^*CA) = \rho((C^*B)^*A) \\ &= (\pi_\omega(C)^*\pi_\omega(B)\Omega_\omega, T\pi_\omega(A)\Omega_\omega) = (\pi_\omega(B)\Omega_\omega, \pi_\omega(C)T\pi_\omega(A)\Omega_\omega) \end{aligned} \quad (4.96)$$

i. e.

$$\forall C \in \mathcal{A} : T\pi_\omega(C) = \pi_\omega(C)T \quad (4.97)$$

or $T \in \pi_\omega(\mathcal{A})'$ and $(\mathcal{H}_\omega, \pi_\omega)$ is reducible.

OTOH, assume that $(\mathcal{H}_\omega, \pi_\omega)$ is reducible. Then there is a $T \in \pi_\omega(\mathcal{A})'$ with $T \not\equiv \lambda \mathbf{1}_{\mathcal{H}_\omega}$. Since $(\pi_\omega(\mathcal{A}))^* = \pi_\omega(\mathcal{A})$, also T^* , $T + T^*$ and $(T - T^*)/i$ lie in the commutant $\pi_\omega(\mathcal{A})'$. Consequently, there is a self-adjoint $S \in \pi_\omega(\mathcal{A})'$ with $S \not\equiv \lambda \mathbf{1}_{\mathcal{H}_\omega}$ and a corresponding spectral projection $P \in \pi_\omega(\mathcal{A})'$ with $0 < P < \mathbf{1}_{\mathcal{H}_\omega}$. This allows to define a $\rho : \mathcal{A} \rightarrow \mathbf{C}$ via

$$\rho(A) = (P\Omega_\omega, \pi_\omega(A)\Omega_\omega) . \quad (4.98)$$

This ρ is positive, because

$$\begin{aligned} \rho(A^*A) &= (P\Omega_\omega, \pi_\omega(A^*A)\Omega_\omega) \\ &= (\pi_\omega(A)P\Omega_\omega, \pi_\omega(A)\Omega_\omega) = (P\pi_\omega(A)\Omega_\omega, P\pi_\omega(A)\Omega_\omega) \geq 0 . \end{aligned} \quad (4.99)$$

Also

$$\omega(A^*A) - \rho(A^*A) = (\pi_\omega(A)\Omega_\omega, (\mathbf{1} - P)\pi_\omega(A)\Omega_\omega) \geq 0 \quad (4.100)$$

and ω majorizes ρ . But ρ can not be a multiple of ω , because P is not a multiple of $\mathbf{1}_{\mathcal{H}_\omega}$ and therefore ω is not pure. This shows that 1) follows from 2).

The correspondence between positive functionals and positive operators has been shown by construction *en passant*. \square

Note that this relation of purity and irreducibility applies only to the GNS representation. One can realize an impure state with a density matrix in an irreducible representation, but this is *not* the GNS representation associated with this state.

—5—

QUANTUM MECHANICS

5.1 Weyl Systems

Definition 5.1. A *symplectic vector space* (V, Θ) is a vector space with V a real vector space and $\Theta : V \times V \rightarrow \mathbf{R}$ a non-degenerate antisymmetric bilinear map, i. e. $\forall v, u, w \in V, \alpha, \beta \in \mathbf{R}$

$$\Theta(v, u) = -\Theta(u, v) \quad (5.1a)$$

$$\Theta(v, \alpha u + \beta w) = \alpha \Theta(v, u) + \beta \Theta(v, w) \quad (5.1b)$$

$$(\forall v \in V : \Theta(v, u) = 0) \Rightarrow u = 0. \quad (5.1c)$$

If the dimension of V is finite, it is necessarily even.

Example 5.2 (canonical phase space). Consider a flat n -dimensional configuration space $q \in \mathbf{R}^n$ and the corresponding phase space

$$z = (z_1, \dots, z_{2n}) = (q, p) = (q_1, \dots, q_n, p_1, \dots, p_n) \in \mathbf{R}^{2n} = V. \quad (5.2)$$

The Poisson brackets $(i, j = 1, \dots, n)$

$$\{q_i, p_j\} = \delta^{ij} \quad (5.3a)$$

$$\{q_i, q_j\} = 0 \quad (5.3b)$$

$$\{p_i, p_j\} = 0 \quad (5.3c)$$

define a symplectic structure $(\alpha, \beta = 1, \dots, 2n)$

$$\{z_\alpha, z_\beta\} = \theta^{\alpha\beta} \quad (5.4)$$

with the antisymmetric $2n \times 2n$ -matrix

$$\theta = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}. \quad (5.5)$$

This corresponds to a symplectic form

$$\begin{aligned} \Theta : \mathbf{R}^{2n} \times \mathbf{R}^{2n} &\rightarrow \mathbf{R} \\ \left(\underbrace{(q, p)}_z, \underbrace{(q', p')}_{z'} \right) &\mapsto \sum_{\alpha, \beta=1}^{2n} z_\alpha \theta^{\alpha\beta} z'_\beta = \sum_{i=1}^n (q_i p'_i - p_i q'_i). \end{aligned} \quad (5.6)$$

Example 5.3 (complex vector space). Using the canonical identification

$$\mathbf{C}^n \ni (x_1 + iy_1, \dots, x_n + iy_n) = (x_1, \dots, x_n, y_1, \dots, y_n) \in \mathbf{R}^{2n} \quad (5.7)$$

the preceding example is equivalent to

$$\begin{aligned} \Theta : \mathbf{C}^n \times \mathbf{C}^n &\rightarrow \mathbf{R} \\ (z, z') &\mapsto \sum_{i=1}^n \operatorname{Im} \bar{z}_i z'_i. \end{aligned} \quad (5.8)$$

Example 5.4 (Peierls brackets). Consider the vector space $\mathcal{S}_m(\mathbf{R}^{n+1})$ of (smooth) solutions $\phi : \mathbf{R} \times \mathbf{R}^n \rightarrow \mathbf{R}$ of the *Klein-Gordon equation* with mass m

$$\left(\frac{\partial^2}{\partial x_0^2} - \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} + m^2 \right) \phi(x) = (\square + m^2) \phi(x) = 0. \quad (5.9)$$

that fall off “sufficiently fast” at “spatial infinity”, i. e. $|x_{1,\dots,n}| \rightarrow \infty$. The *Peierls bracket*

$$\begin{aligned} \langle \cdot, \cdot \rangle : \mathcal{S}_m(\mathbf{R}^{n+1}) \times \mathcal{S}_m(\mathbf{R}^{n+1}) &\rightarrow \mathbf{R} \\ (\phi, \psi) &\mapsto \langle \phi, \psi \rangle = \langle \phi, \psi \rangle_t \end{aligned} \quad (5.10)$$

with

$$\langle \phi, \psi \rangle_t = \int_{x_0=t} d^n x \left(\phi(x) \frac{\partial \psi}{\partial x_0}(x) - \frac{\partial \phi}{\partial x_0}(x) \psi(x) \right) \quad (5.11)$$

is, due to the Klein-Gordon equation and falloff at spatial infinity, independent of t and can be used to turn $\mathcal{S}_m(\mathbf{R}^{n+1})$ into the symplectic vector space $(\mathcal{S}_m(\mathbf{R}^{n+1}), \langle \cdot, \cdot \rangle)$. The nondegeneracy can be shown by considering the initial conditions for the Klein-Gordon equation (\rightarrow *exercise*).

Definition 5.5. A *Weyl system* (\mathcal{A}, W) of a symplectic vector space (V, Θ) is a C^* -algebra \mathcal{A} together with a map $W : V \rightarrow \mathcal{A}$, such that $\forall \phi, \psi \in V$

$$W(0) = \mathbf{1} \quad (5.12a)$$

$$(W(\phi))^* = W(-\phi) \quad (5.12b)$$

$$W(\phi)W(\psi) = e^{-\frac{i}{2}\Theta(\phi, \psi)} W(\phi + \psi). \quad (5.12c)$$

Note that we don't require W to be continuous. In fact, we don't even assume V to have a topology.

Example 5.6 (one particle *quantum mechanics* in one dimension). Consider $V = \mathbf{R}^2$ with

$$\begin{aligned} \Theta : V \times V &\rightarrow \mathbf{R} \\ ((\xi_1, \eta_1), (\xi_2, \eta_2)) &\mapsto \eta_1 \xi_2 - \xi_1 \eta_2. \end{aligned} \quad (5.13)$$

Then the composition law in the corresponding Weyl system is

$$W(\xi_1, \eta_1)W(\xi_2, \eta_2) = e^{\frac{i}{2}(\xi_1 \eta_2 - \eta_1 \xi_2)} W(\xi_1 + \xi_2, \eta_1 + \eta_2). \quad (5.14)$$

The connection with quantum mechanics is to *formally* write

$$W(\xi, \eta) = e^{i\xi p + i\eta x} \quad (5.15)$$

with the *unbounded* hermitian operators

$$x = x^* \quad (5.16a)$$

$$p = p^* \quad (5.16b)$$

$$[x, p] = i \quad (5.16c)$$

$$[x, x] = [p, p] = 0 \quad (5.16d)$$

and to use

$$e^A e^B = e^{A+B + \frac{1}{2}[A, B]} = e^{A+B} e^{\frac{1}{2}[A, B]} \quad (5.17)$$

for $[A, [A, B]] = [[A, B], B] = 0$. Then we obtain

$$\begin{aligned} W(\xi_1, \eta_1)W(\xi_2, \eta_2) &= e^{i\xi_1 p + i\eta_1 x} e^{i\xi_2 p + i\eta_2 x} \\ &= e^{i\xi_1 p + i\eta_1 x + i\xi_2 p + i\eta_2 x} e^{-\frac{i}{2}[\xi_1 p + \eta_1 x, \xi_2 p + \eta_2 x]} = e^{i(\xi_1 + \xi_2)p + i(\eta_1 + \eta_2)x} e^{\frac{i}{2}(\xi_1 \eta_2 - \eta_1 \xi_2)} \\ &= e^{\frac{i}{2}(\xi_1 \eta_2 - \eta_1 \xi_2)} W(\xi_1 + \xi_2, \eta_1 + \eta_2) \end{aligned} \quad (5.18)$$

i. e., up to domain issues for unbounded operators, the *canonical commutation relations* (5.16) are equivalent to the *Weyl form of the canonical commutation relations* (5.14).

Definition 5.7. The algebra \mathcal{A}_W generated by the $W(\xi, \eta)$, i. e. the completion of the linear span of the $W(\xi, \eta)$ is called the *Weyl algebra*.

Example 5.8 (n particle quantum mechanics). Consider $V = \mathbf{R}^{2n}$ with

$$\begin{aligned} \Theta : V \times V &\rightarrow \mathbf{R} \\ ((\xi_1, \eta_1), (\xi_2, \eta_2)) &\mapsto \sum_{i=1}^n (\eta_{1,i} \xi_{2,i} - \xi_{1,i} \eta_{2,i}). \end{aligned} \quad (5.19)$$

Then the composition law in the corresponding Weyl system is

$$W(\xi_1, \eta_1)W(\xi_2, \eta_2) = e^{\frac{i}{2} \sum_{i=1}^n (\xi_{1,i}\eta_{2,i} - \eta_{1,i}\xi_{2,i})} W(\xi_1 + \xi_2, \eta_1 + \eta_2). \quad (5.20)$$

The connection with quantum mechanics is again to *formally* write

$$W(\xi, \eta) = \exp \left(i \sum_{i=1}^n (\xi_i p_i + \eta_i x_i) \right). \quad (5.21)$$

Example 5.9 (general Weyl system). Let $\mathcal{H} = L^2(V, \mathbf{C})$ be the Hilbert space of all functions $f : V \rightarrow \mathbf{C}$ that are square integrable wrt the counting measure¹, i. e. $f \in \mathcal{H}$, iff f vanishes everywhere except for countably many points $\phi \in V$ and

$$\|f\|_{\mathcal{H}}^2 = \sum_{\phi \in V} |f(\phi)|^2 < \infty. \quad (5.22)$$

The inner product on \mathcal{H} is then accordingly

$$\langle f, g \rangle_{\mathcal{H}} = \sum_{\phi \in V} \overline{f(\phi)} g(\phi). \quad (5.23)$$

Let $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ be the C^* -algebra of bounded operators on \mathcal{H} and define

$$\begin{aligned} W : V &\rightarrow \mathcal{A} \\ \phi &\mapsto W(\phi) \end{aligned} \quad (5.24a)$$

where the action of $W(\phi) : \mathcal{H} \rightarrow \mathcal{H}$ is given by

$$(W(\phi)f)(\psi) = e^{\frac{i}{2}\Theta(\phi,\psi)} f(\phi + \psi). \quad (5.24b)$$

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It's easy to check (5.12): each $W(\phi)$ is obviously bounded

$$\forall \phi \in V, f \in \mathcal{H} : \|W(\phi)f\| = \|f\| \quad (5.25)$$

and (5.12a) is also obvious

$$\forall f \in \mathcal{H} : (W(0)f)(\psi) = f(\psi). \quad (5.26)$$

¹Note that everything works for $L^2(V, \mathbf{C})$ with more general measures, including the Lebesgue measuer, except for statement 4) of theorem 5.10, whose proof uses an induction argument.

Verifying (5.12b) requires a short calculation

$$\begin{aligned}
\forall \phi \in V, f, g \in \mathcal{H} : \langle f, (W(\phi))^* g \rangle_{\mathcal{H}} &= \langle W(\phi)f, g \rangle_{\mathcal{H}} \\
&= \sum_{\psi \in V} \overline{(W(\phi)f)(\psi)} g(\psi) = \sum_{\psi \in V} \overline{e^{\frac{i}{2}\Theta(\phi, \psi)} f(\phi + \psi)} g(\psi) \\
&\stackrel{\chi = \phi + \psi}{=} \sum_{\chi \in V} \overline{e^{\frac{i}{2}\Theta(\phi, \chi - \phi)} f(\chi)} g(\chi - \phi) = \sum_{\chi \in V} \overline{f(\chi)} e^{\frac{i}{2}\Theta(-\phi, \chi)} g(\chi - \phi) \\
&= \sum_{\chi \in V} \overline{f(\chi)} (W(-\phi)g)(\chi) = \langle f, W(-\phi)g \rangle_{\mathcal{H}} \quad (5.27)
\end{aligned}$$

similarly for (5.12c)

$$\begin{aligned}
\forall \phi, \psi, \chi \in V, f \in \mathcal{H} : ((W(\phi)W(\psi))f)(\chi) &= (W(\phi)(W(\psi)f))(\chi) \\
&= e^{\frac{i}{2}\Theta(\phi, \chi)} (W(\psi)f)(\phi + \chi) = e^{\frac{i}{2}\Theta(\phi, \chi)} e^{\frac{i}{2}\Theta(\psi, \phi + \chi)} f(\psi + \phi + \chi) \\
&= e^{\frac{i}{2}\Theta(\psi, \phi)} e^{\frac{i}{2}\Theta(\phi + \psi, \chi)} f(\psi + \phi + \chi) = e^{-\frac{i}{2}\Theta(\phi, \psi)} (W(\phi + \psi)f)(\chi) \\
&= \left(e^{-\frac{i}{2}\Theta(\phi, \psi)} W(\phi + \psi)f \right)(\chi). \quad (5.28)
\end{aligned}$$

Theorem 5.10. *Let (\mathcal{A}, W) be a Weyl system of a symplectic vector space (V, Θ) , then*

1. $W(\phi)$ is unitary for all $\phi \in V$,
2. $\|W(\phi) - W(\psi)\| = 2$ for all $\phi \neq \psi \in V$,
3. \mathcal{A} is not separable, unless $V = \{0\}$,
4. the family $\{W(\phi)\}_{\phi \in V}$ is linearly independent.

Proof.

1. Obviously

$$(W(\phi))^* W(\phi) = W(-\phi)W(\phi) = \mathbf{1} \quad (5.29a)$$

$$W(\phi)(W(\phi))^* = W(\phi)W(-\phi) = \mathbf{1}. \quad (5.29b)$$

2. $\forall \phi, \chi \in V$:

$$\begin{aligned}
W(\chi)W(\phi)(W(\chi))^{-1} &= W(\chi)W(\phi)W(-\chi) \\
&= e^{-\frac{i}{2}\Theta(\chi, \phi)} W(\chi + \phi)W(-\chi) = e^{-\frac{i}{2}\Theta(\chi, \phi)} e^{-\frac{i}{2}\Theta(\phi, -\chi)} W(\phi)
\end{aligned}$$

$$= e^{-i\Theta(\chi, \phi)} W(\phi). \quad (5.30)$$

Note that $z\mathbf{1} - A$ is invertible, iff $U(z\mathbf{1} - A)U^{-1} = z\mathbf{1} - UAU^{-1}$ is invertible for invertible U . Therefore the spectrum is invariant under similarity transformations

$$\sigma(A) = \sigma(UAU^{-1}) \quad (5.31)$$

and (5.30) implies

$$\sigma(W(\phi)) = \sigma(W(\chi)W(\phi)(W(\chi))^{-1}) = e^{-i\Theta(\chi, \phi)} \sigma(W(\phi)). \quad (5.32)$$

If $\phi \neq 0$ and Θ is non degenerate, there is a χ such that $V \ni \phi \mapsto \Theta(\chi, \phi)$ is onto \mathbf{R} and $V \ni \phi \mapsto e^{\frac{i}{2}\Theta(\chi, \phi)}$ is onto $S^1 = \{z \in \mathbf{C} : |z| = 1\}$. Thus, for $\phi \neq 0$, the spectrum of $W(\phi)$ is invariant under arbitrary rotations $z \mapsto e^{i\alpha}z$. Since $\sigma(W(\phi)) \neq \emptyset$ by theorem 2.11 and, by unitarity, $\sigma(W(\phi)) \subseteq S^1$ we conclude that

$$\forall \phi \neq 0 : \sigma(W(\phi)) = S^1. \quad (5.33)$$

Therefore

$$\forall \phi \neq \psi : \sigma\left(e^{\frac{i}{2}\Theta(\psi, \phi)} W(\phi - \psi) - \mathbf{1}\right) = S^1 - 1 \quad (5.34)$$

and since $e^{\frac{i}{2}\Theta(\psi, \phi)} W(\phi - \psi) - \mathbf{1}$ is normal, the norm is given by the spectral radius

$$\forall \phi \neq \psi : \left\| e^{\frac{i}{2}\Theta(\psi, \phi)} W(\phi - \psi) - \mathbf{1} \right\| = \rho\left(e^{\frac{i}{2}\Theta(\psi, \phi)} W(\phi - \psi) - \mathbf{1}\right) = 2. \quad (5.35)$$

Using

$$\begin{aligned} W(\phi) - W(\psi) &= W(\psi)(W(-\psi)W(\phi) - \mathbf{1}) \\ &= W(\psi)\left(e^{\frac{i}{2}\Theta(\psi, \phi)} W(\phi - \psi) - \mathbf{1}\right), \end{aligned} \quad (5.36)$$

we obtain the desired result

$$\begin{aligned} \|W(\phi) - W(\psi)\|^2 &= \|(W(\phi) - W(\psi))^*(W(\phi) - W(\psi))\| \\ &= \left\| \left(e^{\frac{i}{2}\Theta(\psi, \phi)} W(\phi - \psi) - \mathbf{1}\right)^* (W(\psi))^* W(\psi) \left(e^{\frac{i}{2}\Theta(\psi, \phi)} W(\phi - \psi) - \mathbf{1}\right) \right\| \\ &= \left\| \left(e^{\frac{i}{2}\Theta(\psi, \phi)} W(\phi - \psi) - \mathbf{1}\right)^* \left(e^{\frac{i}{2}\Theta(\psi, \phi)} W(\phi - \psi) - \mathbf{1}\right) \right\| \\ &= \left\| e^{\frac{i}{2}\Theta(\psi, \phi)} W(\phi - \psi) - \mathbf{1} \right\|^2 = 4. \end{aligned} \quad (5.37)$$

3. Since $\|W(\phi) - W(\psi)\| = 2$, the open balls of radius 1 that are centered at $\phi \in V$ are a collection of disjoint open sets that is *not* countable.
4. Assume that there is a $\{\phi_i\}_{i=1,\dots,n}$ such that $\phi_i \neq \phi_j$ for $i \neq j$ and

$$\sum_{i=1}^n \alpha_i W(\phi_i) = 0. \quad (5.38)$$

We will show by induction on n that $\alpha_i = 0$. The case $n = 1$ is trivial, because all $W(\phi)$ are unitary and thus never null. For the induction step², assume **WLOG** that the $\{W(\phi_i)\}_{i=1,n-1}$ are linearly independent and $\alpha_n \neq 0$. Then

$$W(\phi_n) = - \sum_{i=1}^{n-1} \frac{\alpha_i}{\alpha_n} W(\phi_i) \quad (5.39)$$

and for all $\psi \in V$

$$\begin{aligned} e^{-i\Theta(\psi, \phi_n)} \left(- \sum_{i=1}^{n-1} \frac{\alpha_i}{\alpha_n} W(\phi_i) \right) &\stackrel{(5.39)}{=} e^{-i\Theta(\psi, \phi_n)} W(\phi_n) \\ &\stackrel{(5.30)}{=} W(\psi) W(\phi_n) W(-\psi) \\ &\stackrel{(5.39)}{=} - \sum_{i=1}^{n-1} \frac{\alpha_i}{\alpha_n} W(\psi) W(\phi_i) W(-\psi) \stackrel{(5.30)}{=} - \sum_{i=1}^{n-1} \frac{\alpha_i}{\alpha_n} e^{-i\Theta(\psi, \phi_i)} W(\phi_i) \end{aligned} \quad (5.40)$$

i. e.

$$\sum_{i=1}^{n-1} \alpha_i W(\phi_i) = \sum_{i=1}^{n-1} \alpha_i e^{i\Theta(\psi, \phi_n - \phi_i)} W(\phi_i) \quad (5.41)$$

or, by the induction hypothesis that the $\{W(\phi_i)\}_{i=1,\dots,n-1}$ are independent

$$\forall i = 1, \dots, n-1 : \alpha_i = \alpha_i e^{i\Theta(\psi, \phi_n - \phi_i)}. \quad (5.42)$$

If $\alpha_i \neq 0$ for $i < n$, then

$$\forall \psi \in V : \Theta(\psi, \phi_n - \phi_i) = 0 \pmod{2\pi}. \quad (5.43)$$

By linearity

$$\forall \psi \in V : \Theta(\psi, \phi_n - \phi_i) = 0 \quad (5.44)$$

²NB: the following computation is clearer than the one adapted from [BGP07], which had been presented in the lecture of Dec. 5, 2018

and since Θ is non-degenerate we find the contradiction

$$\phi_n = \phi_i. \quad (5.45)$$

Thus $\forall i = 1, \dots, n : \alpha_i = 0$.

□

5.2 The Stone-von Neumann Theorem

In the following, it will sometimes be convenient to use in place of the $W(\xi, \eta)$ from example 5.6 Weyl's

$$U(\eta) = W(0, \eta) \quad (5.46a)$$

$$V(\xi) = W(\xi, 0), \quad (5.46b)$$

which form two abelian one-parameter groups of unitary elements of \mathcal{A}_W

$$U(\eta)U(\eta') = U(\eta + \eta') \quad (5.47a)$$

$$V(\xi)V(\xi') = V(\xi + \xi'). \quad (5.47b)$$

The general $W(\xi, \eta)$ can of course be expressed by $U(\eta)$ and $V(\xi)$ since

$$U(\eta)V(\xi) = W(0, \eta)W(\xi, 0) = W(\xi, \eta)e^{-\frac{i}{2}\eta\xi} \quad (5.48a)$$

$$V(\xi)U(\eta) = W(\xi, 0)W(0, \eta) = W(\xi, \eta)e^{\frac{i}{2}\xi\eta}, \quad (5.48b)$$

i. e.

$$W(\xi, \eta) = e^{-\frac{i}{2}\eta\xi}V(\xi)U(\eta) = e^{\frac{i}{2}\eta\xi}U(\eta)V(\xi). \quad (5.49)$$

The analogous notation can also be used for the n degree of freedom in example 5.8.

Definition 5.11 (*regular representation*). A representation (\mathcal{H}, π) of the Weyl algebra \mathcal{A}_W on a separable Hilbert space \mathcal{H} is called *regular*, if $\mathbf{R} \ni \xi \mapsto \pi(V(\xi)) = \pi(W(\xi, 0))$ and $\mathbf{R} \ni \eta \mapsto \pi(U(\eta)) = \pi(W(0, \eta))$ are strongly continuous in ξ and η , respectively.

Theorem 5.12 (*Stone-von Neumann*). All regular irreducible representations of the Weyl algebra \mathcal{A}_W of a finite dimensional symplectic vector space are unitarily equivalent.

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Proof. Starting from the Fock state $\omega_F : \mathcal{A}_W \rightarrow \mathbf{C}$, defined by

$$\omega_F(W(\xi, \eta)) = \exp\left(-\frac{\xi^2 + \eta^2}{4}\right) \quad (5.50)$$

and linearity, we construct the corresponding GNS representation $(\mathcal{H}_F, \pi_F, \Omega_F)$ with

$$(\Omega_F, \pi_F(W(\xi, \eta))\Omega_F) = \omega_F(W(\xi, \eta)). \quad (5.51)$$

Now consider another regular representation π and compute the integral

$$P_\pi = \int \frac{d\xi d\eta}{2\pi} e^{-\frac{\xi^2 + \eta^2}{4}} \pi(W(\xi, \eta)). \quad (5.52)$$

This integral exists as a limit of Riemann sums in the strong operator topology, because $(\xi, \eta) \mapsto e^{-\frac{\xi^2 + \eta^2}{4}} \in L^1(\mathbf{R}^2)$ and $\pi(W(\xi, \eta))$ is both bounded and continuous in ξ and η . P_π is obviously self-adjoint

$$P_\pi^* = \int \frac{d\xi d\eta}{2\pi} e^{-\frac{\xi^2 + \eta^2}{4}} \pi(W(-\xi, -\eta)) = P_\pi. \quad (5.53)$$

It is not obvious that P_π doesn't vanish. However, if this was the case, then using

$$\begin{aligned} W(-\xi', -\eta')W(\xi, \eta)W(\xi', \eta') &= e^{-\frac{i}{2}(\xi'\eta - \eta'\xi)} W(\xi - \xi', \eta - \eta')W(\xi', \eta') \\ &= e^{-\frac{i}{2}(\xi'\eta - \eta'\xi)} e^{\frac{i}{2}(\xi\eta' - \eta\xi')} W(\xi, \eta) = e^{i\xi\eta' - i\eta\xi'} W(\xi, \eta) \end{aligned} \quad (5.54)$$

(that's (5.30) again), we find

$$\begin{aligned} 0 &= \pi(W(-\xi', -\eta'))P_\pi\pi(W(\xi', \eta')) \\ &= \int \frac{d\xi d\eta}{2\pi} e^{-\frac{\xi^2 + \eta^2}{4}} \pi(W(-\xi', -\eta')W(\xi, \eta)W(\xi', \eta')) \\ &= \int \frac{d\xi d\eta}{2\pi} e^{-\frac{\xi^2 + \eta^2}{4}} \pi(W(\xi, \eta)) e^{i\xi\eta' - i\eta\xi'}, \end{aligned} \quad (5.55)$$

i. e. that the Fourier transform of all matrix elements of

$$e^{-\frac{\xi^2 + \eta^2}{4}} \pi(W(\xi, \eta)) \quad (5.56)$$

would vanish, which would imply $\pi(W(\xi, \eta)) = 0$. Also, using

$$\begin{aligned}
W(\xi_1, \eta_1)W(\xi, \eta)W(\xi_2, \eta_2) &= e^{\frac{i}{2}(\xi_1\eta - \eta_1\xi)}W(\xi + \xi_1, \eta + \eta_1)W(\xi_2, \eta_2) \\
&= e^{\frac{i}{2}(\xi_1\eta - \eta_1\xi)}e^{\frac{i}{2}((\xi + \xi_1)\eta_2 - (\eta + \eta_1)\xi_2)}W(\xi + \xi_1 + \xi_2, \eta + \eta_1 + \eta_2) \\
&= e^{\frac{i}{2}(\xi_1\eta + \xi\eta_2 + \xi_1\eta_2 - \eta_1\xi - \eta\xi_2 - \eta_1\xi_2)}W(\xi + \xi_1 + \xi_2, \eta + \eta_1 + \eta_2) \quad (5.57)
\end{aligned}$$

we find by a straightforward computation of two Gaussian integrals (\rightarrow *exercise*)

$$P_\pi \pi(W(\xi, \eta))P_\pi = e^{-\frac{\xi^2 + \eta^2}{4}} P_\pi. \quad (5.58)$$

In particular from

$$P_\pi^2 = P_\pi \pi(W(0, 0))P_\pi = P_\pi, \quad (5.59)$$

we see that $P_\pi \neq 0$ is an orthogonal projection. Thus there must be a $\Omega_0 \in \mathcal{H}_\pi$ with

$$P_\pi \Omega_0 = \Omega_0 \quad (5.60a)$$

$$\|\Omega_0\| = 1 \quad (5.60b)$$

and we can compute

$$\begin{aligned}
(\Omega_0, \pi(W(\xi, \eta))\Omega_0) &= (\Omega_0, P_\pi \pi(W(\xi, \eta))P_\pi \Omega_0) \\
&\stackrel{(5.58)}{=} e^{-\frac{\xi^2 + \eta^2}{4}} \underbrace{(\Omega_0, P_\pi \Omega_0)}_{\|\Omega_0\|^2} = e^{-\frac{\xi^2 + \eta^2}{4}} = \omega_F(W(\xi, \eta)). \quad (5.61)
\end{aligned}$$

This shows that $\Omega_0 = \Omega_\pi$ and we have already shown in theorem 4.7 that all GNS representations corresponding to the same state are unitarily equivalent. \square

We have proven the theorem for one degree of freedom or $V = \mathbf{R} \times \mathbf{R}$, but the same argument works for any *finite* number of degrees of freedom. However, it breaks down for an infinite number of degrees of freedom, because then the integral in (5.52) doesn't exist straightforwardly.

Remark 5.13. P_π projects on a one-dimensional subspace.

Proof. Assume

$$\exists \Psi \in \mathcal{H}_\pi : (\Psi, \Omega_0) = 0 \wedge P_\pi \Psi = \Psi \neq 0, \quad (5.62)$$

then

$$\begin{aligned}
\forall \xi, \eta \in \mathbf{R} : (\Psi, \pi(W(\xi, \eta))\Omega_0) &= (P_\pi \Psi, \pi(W(\xi, \eta))P_\pi \Omega_0) \\
&= e^{-\frac{\xi^2 + \eta^2}{4}} (\Psi, P_\pi \Omega_0) = e^{-\frac{\xi^2 + \eta^2}{4}} (\Psi, \Omega_0) = 0 \quad (5.63)
\end{aligned}$$

and thus

$$\forall A \in \mathcal{A}_W : (\Psi, \pi(A)\Omega_0) = 0. \quad (5.64)$$

Since every $\Psi \in \mathcal{H}_\pi$ is cyclic in an irreducible representation, we conclude that $\Psi = 0$. \square

Note that we needed the strong continuity of the representation only to show that the integral (5.52) exists. We could have relaxed this condition to measurability of

$$\xi \mapsto (\Psi, \pi(V(\xi))\Phi) \quad (5.65a)$$

$$\eta \mapsto (\Psi, \pi(U(\eta))\Phi) \quad (5.65b)$$

for all $\Psi, \Phi \in \mathcal{H}$. However there is a theorem by von Neumann (see, e. g., [RS80], theorem VIII.9, p. 268), that weak measurability implies strong continuity. Thus we may demand strong continuity, **WLOG**.

Using (5.48), we can compute the Fock state of a product of $U(\eta)$ and $V(\xi)$:

$$\omega_F(U(\eta)V(\xi)) = e^{-\frac{i}{2}\xi\eta}\omega_F(W(\xi, \eta)) = e^{-\frac{\xi^2+\eta^2}{4}-\frac{i}{2}\xi\eta}. \quad (5.66)$$

Computing the partial derivatives from this explicit expression

$$\frac{\partial}{\partial \xi}\omega_F(U(\eta)V(\xi)) = -\frac{i\eta + \xi}{2}\omega_F(U(\eta)V(\xi)) \quad (5.67a)$$

$$\frac{\partial}{\partial \eta}\omega_F(U(\eta)V(\xi)) = -\frac{i\xi + \eta}{2}\omega_F(U(\eta)V(\xi)), \quad (5.67b)$$

we can derive the differential equation

$$\left(-i\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} + \eta\right)\omega_F(U(\eta)V(\xi)) = 0. \quad (5.68)$$

Since $\eta \mapsto \pi_F(U(\eta))$ and $\xi \mapsto \pi_F(V(\xi))$ are strongly continuous, Stone's Theorem (see, e. g., [RS80], theorem VIII.8, p. 266), guarantees the existence of self-adjoint operators X and P that are in general unbounded, but defined on dense subdomains of $D_X, D_Y \subseteq \mathcal{H}_\pi$

$$X : D_X \rightarrow \mathcal{H}_\pi \quad (5.69a)$$

$$P : D_P \rightarrow \mathcal{H}_\pi \quad (5.69b)$$

with

$$\pi_F(U(\eta)) = e^{i\eta X} \quad (5.70a)$$

$$\pi_F(V(\xi)) = e^{i\xi P}. \quad (5.70b)$$

Thus we can write

$$-i \frac{\partial}{\partial \xi} \pi_F(V(\xi)) = P \pi_F(V(\xi)) = \pi_F(V(\xi)) P \quad (5.71a)$$

$$\frac{\partial}{\partial \eta} \pi_F(U(\eta)) = iX \pi_F(U(\eta)) = i \pi_F(U(\eta)) X \quad (5.71b)$$

and find

$$\left(-i \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} + \eta \right) \pi_F(U(\eta)) \pi_F(V(\xi)) = \pi_F(U(\eta)) (P + iX + \eta) \pi_F(V(\xi)) \quad (5.72)$$

i. e., from (5.68),

$$\begin{aligned} 0 &= \left(-i \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} + \eta \right) (\Omega_F, \pi_F(U(\eta)V(\xi)) \Omega_F) \\ &= (\Omega_F, \pi_F(U(\eta)) (P + iX + \eta) \pi_F(V(\xi)) \Omega_F) . \end{aligned} \quad (5.73)$$

Since

$$X = -i \lim_{\eta \rightarrow 0} \frac{\partial}{\partial \eta} \pi_F(U(\eta)) \quad (5.74a)$$

$$P = -i \lim_{\xi \rightarrow 0} \frac{\partial}{\partial \xi} \pi_F(V(\xi)) . \quad (5.74b)$$

we can compute

$$\begin{aligned} &X \pi_F(W(\xi, \eta)) - \pi_F(W(\xi, \eta)) X \\ &= -i \lim_{\eta' \rightarrow 0} \frac{\partial}{\partial \eta'} (\pi_F(W(0, \eta')) \pi_F(W(\xi, \eta)) - \pi_F(W(\xi, \eta)) \pi_F(W(0, \eta'))) \\ &= -i \lim_{\eta' \rightarrow 0} \frac{\partial}{\partial \eta'} \left(\pi_F(W(\xi, \eta + \eta')) e^{-\frac{i}{2} \xi \eta'} - \pi_F(W(\xi, \eta + \eta')) e^{\frac{i}{2} \xi \eta'} \right) \\ &= -2 \lim_{\eta' \rightarrow 0} \frac{\partial}{\partial \eta'} \left(\pi_F(W(\xi, \eta + \eta')) \sin \frac{\xi \eta'}{2} \right) \\ &= -2 \lim_{\eta' \rightarrow 0} \left(\sin \frac{\xi \eta'}{2} \frac{\partial}{\partial \eta'} \pi_F(W(\xi, \eta + \eta')) + \pi_F(W(\xi, \eta + \eta')) \frac{\xi}{2} \cos \frac{\xi \eta'}{2} \right) \\ &= -\xi \pi_F(W(\xi, \eta)) \end{aligned} \quad (5.75)$$

and analogously

$$\begin{aligned} &P \pi_F(W(\xi, \eta)) - \pi_F(W(\xi, \eta)) P \\ &= -i \lim_{\xi' \rightarrow 0} \frac{\partial}{\partial \xi'} (\pi_F(W(\xi', 0)) \pi_F(W(\xi, \eta)) - \pi_F(W(\xi, \eta)) \pi_F(W(\xi', 0))) \end{aligned}$$

$$\begin{aligned}
&= -i \lim_{\xi' \rightarrow 0} \frac{\partial}{\partial \xi'} \left(\pi_F(W(\xi + \xi', \eta)) e^{\frac{i}{2}\xi'\eta} - \pi_F(W(\xi + \xi', \eta)) e^{-\frac{i}{2}\xi'\eta} \right) \\
&= 2 \lim_{\xi' \rightarrow 0} \frac{\partial}{\partial \xi'} \left(\pi_F(W(\xi + \xi', \eta)) \sin \frac{\xi'\eta}{2} \right) \\
&= 2 \lim_{\xi' \rightarrow 0} \left(\sin \frac{\xi'\eta}{2} \frac{\partial}{\partial \xi'} \pi_F(W(\xi + \xi', \eta)) + \pi_F(W(\xi + \xi', \eta)) \frac{\eta}{2} \cos \frac{\xi'\eta}{2} \right) \\
&= \eta \pi_F(W(\xi, \eta)), \quad (5.76)
\end{aligned}$$

i. e.

$$X \pi_F(W(\xi, \eta)) = \pi_F(W(\xi, \eta))(X - \xi) \quad (5.77a)$$

$$P \pi_F(W(\xi, \eta)) = \pi_F(W(\xi, \eta))(P + \eta). \quad (5.77b)$$

Using this, we can infer from (5.73)

$$\begin{aligned}
0 &= (\Omega_F, \pi_F(U(\eta))(X - iP - i\eta)\pi_F(V(\xi))\Omega_F) \\
&= (\Omega_F, (X - iP)\pi_F(U(\eta))\pi_F(V(\xi))\Omega_F) \\
&= ((X + iP)\Omega_F, \pi_F(U(\eta))\pi_F(V(\xi))\Omega_F) \quad (5.78)
\end{aligned}$$

i. e.

$$\forall \Psi \in \mathcal{H}_\pi : ((X + iP)\Omega_F, \Psi) = 0 \quad (5.79)$$

or

$$(X + iP)\Omega_F = 0, \quad (5.80)$$

which should be familiar from the algebraic solution of the harmonic oscillator

$$H = \frac{1}{2}p^2 + \frac{1}{2}x^2 = a^*a + \frac{1}{2} \quad (5.81)$$

with

$$a = \frac{1}{\sqrt{2}}(x + ip) \quad (5.82)$$

in elementary quantum mechanics.

5.3 Schrödinger Representation & Wave Functions

In the following, we will construct a particular representation (\mathcal{H}_S, π_S) of \mathcal{A}_W . The Hilbert space is $\mathcal{H}_S = L^2(\mathbf{R}, \mathbf{C})$ with the standard Lebesgue measure on \mathbf{R} . And $\pi_S : \mathcal{A}_W \rightarrow \mathcal{L}(\mathcal{H}_S)$ is uniquely defined by the action of the generating elements $\{\pi_S(W(\xi, \eta))\}_{\xi, \eta \in \mathbf{R}}$ on the elements of \mathcal{H}_S :

$$(\pi_S(U(\eta))\psi)(x) = e^{i\eta x}\psi(x) \quad (5.83a)$$

$$(\pi_S(V(\xi))\psi)(x) = \psi(x + \xi) \quad (5.83b)$$

therefore

$$\begin{aligned} (\pi_S(W(\xi, \eta))\psi)(x) &= \left(\left(e^{\frac{i}{2}\eta\xi} \pi_S(U(\eta))\pi_S(V(\xi)) \right) \psi \right) (x) \\ &= e^{\frac{i}{2}\eta\xi} (\pi_S(U(\eta)) (\pi_S(V(\xi))\psi)) (x) = e^{\frac{i}{2}\eta\xi} e^{i\eta x} (\pi_S(V(\xi))\psi) (x) \\ &= e^{i\eta(x+\xi/2)} \psi(x + \xi). \end{aligned} \quad (5.84)$$

This definition satisfies the Weyl relations (5.14) (\rightarrow exercise). Note that is different from the general Weyl system described in example 5.9, where the $\pi(W(\xi, \eta))$ act on functions on \mathbf{R}^2 . Here we have functions on \mathbf{R} .

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Theorem 5.14. *The Schrödinger representation (\mathcal{H}_S, π_S) with (5.83) is irreducible.*

Proof. If (\mathcal{H}_S, π_S) were reducible, there would be a $\mathcal{H}' \subset \mathcal{H}_S$ invariant under $\pi_S(\mathcal{A}_W)$ and a $\phi \in \mathcal{H}_S$ with

$$\begin{aligned} \forall \psi \in \mathcal{H}', \xi, \eta \in \mathbf{R} : 0 &= (\phi, \pi_S(U(\eta)V(\xi))\psi) \\ &= (\pi_S(U(-\eta))\phi, \pi_S(V(\xi))\psi) = \int dx e^{i\eta x} \overline{\phi(x)} \psi(x + \xi), \end{aligned} \quad (5.85)$$

i. e., the Fourier transform of

$$x \mapsto \overline{\phi(x)} \psi(x + \xi) \quad (5.86)$$

vanishes for all $\xi \in \mathbf{R}$. Therefore the intersection of the support of ϕ and the support of ψ shifted by ξ vanishes. Since ξ is arbitrary, and the support of ψ is not empty, the support of ϕ must be empty, i. e. $\phi = 0$. Thus (\mathcal{H}_S, π_S) can't be reducible. \square

Theorem 5.15. *The Schrödinger representation (\mathcal{H}_S, π_S) is strongly continuous as maps $\mathbf{R} \rightarrow \mathcal{L}(L^2(\mathbf{R}, \mathbf{C}))$*

$$\xi \mapsto \pi_S(V(\xi)) \quad (5.87a)$$

$$\eta \mapsto \pi_S(U(\eta)) \quad (5.87b)$$

with (5.83).

Proof. First compute

$$\|\pi_S(U(\eta))\psi - \psi\|^2 = \int dx \underbrace{|\psi(x)|^2 |e^{i\eta x} - 1|^2}_{\leq 4|\psi(x)|^2} \quad (5.88)$$

and observe that both

$$x \mapsto 4|\psi(x)|^2 \in L^1(\mathbf{R}, \mathbf{C}) \quad (5.89)$$

and

$$\lim_{\eta \rightarrow 0} |\psi(x)|^2 |e^{i\eta x} - 1|^2 = 0 \quad (5.90)$$

pointwise. Therefore, by the *Lebesgue dominated convergence theorem*³, we have convergence of the integrand in the $L^1(\mathbf{R}, \mathbf{C})$ topology and therefore

$$\lim_{\eta \rightarrow 0} \|\pi_S(U(\eta))\psi - \psi\|^2 = 0 \quad (5.91)$$

in the $L^2(\mathbf{R}, \mathbf{C})$ topology. In the case of $\pi_S(V(\xi))$, we can use the same argument again for the Fourier transform $k \mapsto \tilde{\psi}(k)$ of $x \mapsto \psi(x)$ to find⁴

$$\lim_{\xi \rightarrow 0} \|\pi_S(V(\xi))\psi - \psi\|^2 = 0. \quad (5.92)$$

□

³E. g. stated as Theorem I.11 on p. 17 of [RSS0]:

Theorem 5.16 (*Lebesgue dominated convergence theorem*). *If $\forall x \in \mathbf{R} : f_n(x) \rightarrow f(x)$ pointwise and $\exists g \in L^1(\mathbf{R}) : \forall x \in \mathbf{R}, \forall n : |f_n(x)| \leq g(x)$, then $f \in L^1(\mathbf{R})$ and $\int dx |f_n(x) - f(x)| \rightarrow 0$.*

⁴NB: the estimate

$$\|\pi_S(V(\xi))\psi - \psi\|^2 = \int dx \underbrace{|\psi(x + \xi) - \psi(x)|^2}_{\leq (|\psi(x + \xi)| + |\psi(x)|)^2}$$

used in the lecture of Dec. 5, 2018 is *not* sufficient for an application of the *Lebesgue dominated convergence theorem*, because it is not uniform in ξ .

However, as already suggested during the lecture, the argument can be salvaged by starting with continuous functions of compact support, for which

$$\forall \xi_0 : \exists \epsilon(\xi_0) : \forall 0 \leq \xi < \xi_0 : \forall x : |\psi(x + \xi) - \psi(x)| \leq \epsilon(\xi_0).$$

We can approximate any $\psi \in \mathcal{H}_S$ by a sequence $\{\psi_k\}_{k \in \mathbf{N}}$ of continuous functions of compact support and estimate

$$\|\pi_S(V(\xi))\psi - \psi\| \leq \underbrace{\|\pi_S(V(\xi))\psi - \pi_S(V(\xi))\psi_k\|}_{=\|\psi - \psi_k\| \rightarrow 0} + \underbrace{\|\pi_S(V(\xi))\psi_k - \psi_k\|}_{\rightarrow 0} + \underbrace{\|\psi_k - \psi\|}_{\rightarrow 0},$$

using the unitarity of $\pi_S(V(\xi))$.

Having established the strong continuity, we can use Stone's theorem to construct the self-adjoint operators

$$X = -i \lim_{\eta \rightarrow 0} \frac{\partial}{\partial \eta} \pi_S(U(\eta)) \quad (5.93a)$$

$$P = -i \lim_{\xi \rightarrow 0} \frac{\partial}{\partial \xi} \pi_S(V(\xi)). \quad (5.93b)$$

On one hand

$$XP = - \lim_{\xi, \eta \rightarrow 0} \frac{\partial}{\partial \eta} \frac{\partial}{\partial \xi} \pi_S(U(\eta)V(\xi)) \quad (5.94)$$

and on the other

$$\begin{aligned} PX &= - \lim_{\xi, \eta \rightarrow 0} \frac{\partial}{\partial \eta} \frac{\partial}{\partial \xi} \pi_S(V(\xi)U(\eta)) \\ &= - \lim_{\xi, \eta \rightarrow 0} \frac{\partial}{\partial \eta} \frac{\partial}{\partial \xi} (\pi_S(U(\eta)V(\xi))e^{i\eta\xi}) = XP - i\mathbf{1}. \end{aligned} \quad (5.95)$$

Thus

$$XP - PX = [X, P] = i\mathbf{1} \quad (5.96)$$

and X and P can play the role of position and momentum operators on a common dense domain $\mathcal{S}(\mathbf{R}) \subset \mathcal{H}_S$, i. e. the *Schwartz space* of smooth functions that fall off faster than any inverse power. We can therefore use $f(X)$ and $g(P)$ the observables representing functions of position and momentum, respectively.

5.3.1 N Degrees of Freedoms

We can easily extend the Schrödinger representation (\mathcal{H}_S, π_S) to a \mathcal{A}_W with N degrees of freedom. The Hilbert space is now $\mathcal{H}_S = L^2(\mathbf{R}^N, \mathbf{C})$ with the standard Lebesgue measure on \mathbf{R}^N . The homomorphism $\pi_S : \mathcal{A}_W \rightarrow \mathcal{L}(\mathcal{H}_S)$ is uniquely defined by the action of $\{\pi_S(W(\vec{\xi}, \vec{\eta}))\}_{\vec{\xi}, \vec{\eta} \in \mathbf{R}^N}$ on the elements of \mathcal{H}_S by

$$(\pi_S(W(\vec{\xi}, \vec{\eta}))\psi)(\vec{x}) = e^{i\vec{\eta}(\vec{x} + \vec{\xi}/2)}\psi(\vec{x} + \vec{\xi}). \quad (5.97)$$

All regular representations of the Weyl algebra for N d.o.f. are again unitarily equivalent, since the proof given above goes through.

5.3.2 Equivalent Representations

We can generalize the above representation $\pi = \pi_S$ to a two-parameter family of representations $(\mathcal{H}_{\alpha,\beta}, \pi_{\alpha,\beta})$ with $\alpha, \beta \in \mathbf{R}$

$$\pi_{\alpha,\beta}(W(\xi, \eta)) = e^{i(\xi\beta - \eta\alpha)} \pi(W(\xi, \eta)) \quad (5.98)$$

where obviously $\mathcal{H} = \mathcal{H}_{0,0}$ and $\pi = \pi_{0,0}$. In particular

$$(\pi_{\alpha,\beta}(U(\eta))\psi)(x) = e^{i\eta(x-\alpha)} \psi(x) \quad (5.99a)$$

$$(\pi_{\alpha,\beta}(V_\beta(\xi))\psi)(x) = e^{i\xi\beta} \psi(x + \xi) \quad (5.99b)$$

or

$$(\pi_{\alpha,\beta}(W(\xi, \eta))\psi)(x) = e^{i\eta(x+\xi/2)+i(\xi\beta-\eta\alpha)} \psi(x + \xi). \quad (5.100)$$

According to the Stone-von Neumann theorem, all these representations must be unitarily equivalent. This is indeed the case and we can use $\mathcal{H}_{\alpha,\beta} = \mathcal{H}_{0,0} = \mathcal{H}_{0,0}$ and $\pi_{\alpha,\beta} = \pi_{0,0} \circ U_{\alpha,\beta} = \pi \circ U_{\alpha,\beta}$ with the intertwiners

$$U_{\alpha,\beta} = \pi(W(\alpha, \beta)) \in \mathcal{L}(\mathcal{H}), \quad (5.101)$$

because from (5.57)

$$W(-\alpha, -\beta)W(\xi, \eta)W(\alpha, \beta) = e^{i(\beta\xi - \eta\alpha)} W(\xi, \eta) \quad (5.102)$$

and therefore

$$U_{\alpha,\beta}^{-1} \pi(W(\xi, \eta)) U_{\alpha,\beta} = e^{i(\beta\xi - \eta\alpha)} \pi(W(\xi, \eta)) = \pi_{\alpha,\beta}(W(\xi, \eta)). \quad (5.103)$$

5.3.3 Pitfalls on the Circle

The proof of the Stone-von Neumann theorem presented above works equally well for any finite number of degrees of freedom, but fails for an infinite number of **d.o.f.**, because the infinite product of Gaussian integrals in (5.52) is not well defined. However, there are simpler systems, where the Stone-von Neumann theorem doesn't apply.

Consider a particle on the circle $S^1 = \{(x, y) \in \mathbf{R}^2 : x^2 + y^2 = 1\}$. A more convenient parametrization is

$$S^1 = \{(\cos \phi, \sin \phi) \in \mathbf{R}^2 : 0 \leq \phi < 2\pi\}, \quad (5.104)$$

but we must impose periodic boundary conditions for wavefunctions at $\phi \rightarrow 2\pi \cong 0$.

In particular, we must have

$$\forall \psi \in \mathcal{H}_\pi : (\pi(W(\xi, \eta))\psi)(x) = (\pi(W(\xi, \eta))\psi)(x + 2\pi) \quad (5.105)$$

because otherwise $\pi(W(\xi, \eta))$ would destroy periodicity. Thus

$$e^{i\eta(x+\xi/2)}\psi(x+\xi) \stackrel{!}{=} e^{i\eta(x+2\pi+\xi/2)}\psi(x+2\pi+\xi) = e^{i\eta(x+2\pi+\xi/2)}\psi(x+\xi), \quad (5.106)$$

or

$$e^{i\eta(x+\xi/2)} \stackrel{!}{=} e^{i\eta(x+2\pi+\xi/2)} \quad (5.107)$$

i. e. $\eta \in \mathbf{Z}$. Going back from the representation to the algebra, we see that the Weyl-algebra for S^1 is generated by

$$\left\{ \widetilde{W}_\nu(\xi) = W(\xi, \nu) : \xi \in \mathbf{R}, \nu \in \mathbf{Z} \right\}. \quad (5.108)$$

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We can again study the representations $(\mathcal{H}_{\alpha,\beta}, \pi_{\alpha,\beta})$, but this time, there are fewer intertwiners available and we can only construct the representations with $\beta \in \mathbf{Z}$ using

$$U_{\alpha,\beta} = \pi \left(\widetilde{W}_\beta(\alpha) \right), \quad (5.109)$$

while the representations $(\mathcal{H}_{\alpha,\beta}, \pi_{\alpha,\beta})$ with $0 < \beta < 1$ could be inequivalent. The Stone-von Neumann theorem is also of little help, because the conditions are not met and the proof breaks down, since we can't use Gaussian integrals over η to construct a projector. We can however obtain a very simple necessary condition for two representations to be unitarily equivalent from the observation

Lemma 5.17. *The spectrum $\sigma(A)$ of an operator $A \in \mathcal{L}(\mathcal{H})$ is invariant under similarity transformations $A \rightarrow SAS^{-1} \in \mathcal{L}(\mathcal{H}')$ with invertible $S : \mathcal{H} \rightarrow \mathcal{H}'$.*

$$\forall A \in \mathcal{L}(\mathcal{H}) : \sigma_{\mathcal{L}(\mathcal{H})}(A) = \sigma_{\mathcal{L}(\mathcal{H}')} (SAS^{-1}). \quad (5.110)$$

Proof. $S^{-1} : \mathcal{H}' \rightarrow \mathcal{H}$ exists and $B = SAS^{-1} \in \mathcal{L}(\mathcal{H}')$ is well defined. The operator

$$z\mathbf{1} - B = z\mathbf{1} - SAS^{-1} = S(z\mathbf{1} - A)S^{-1} \quad (5.111)$$

is invertible **iff** $z\mathbf{1} - A$ is invertible. Thus the resolvent set and the spectrum agree. \square

Corollary 5.18. *The spectrum of the representatives is the same in two unitarily equivalent representations (\mathcal{H}_1, π_1) and (\mathcal{H}_2, π_2) of a C^* -algebra \mathcal{A} :*

$$\forall A \in \mathcal{A} : \sigma_{\mathcal{L}(\mathcal{H}_1)}(\pi_1(A)) = \sigma_{\mathcal{L}(\mathcal{H}_2)}(\pi_2(A)). \quad (5.112)$$

Proof. The unitary intertwiner $U : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ defines a similarity transformation $\pi_1(A) \rightarrow \pi_2(A) = U\pi_1(A)U^{-1}$. \square

Of course, $z\mathbf{1} - \pi(W(\xi, \eta))$ is invertible, i. e. $z \notin \sigma(\pi(W(\xi, \eta)))$, **iff**

$$e^{i(\beta\xi - \eta\alpha)} z\mathbf{1} - \pi_{\alpha, \beta}(W(\xi, \eta)) = e^{i(\beta\xi - \eta\alpha)} (z\mathbf{1} - \pi(W(\xi, \eta))) \quad (5.113)$$

is invertible, i. e. $e^{i(\beta\xi - \eta\alpha)} z \notin \sigma(\pi_{\alpha, \beta}(W(\xi, \eta)))$. Thus we have shown that

$$\sigma(\pi_{\alpha, \beta}(W(\xi, \eta))) = e^{i(\beta\xi - \eta\alpha)} \sigma(\pi(W(\xi, \eta))) \quad (5.114)$$

and in particular

$$\sigma(\pi_{\alpha, \beta}(W(\xi, 0))) = e^{i\beta\xi} \sigma(\pi(W(\xi, 0))). \quad (5.115)$$

If we can compute $\sigma(\pi(W(\xi, 0)))$ and show that is a *strict* subset of S^1 *not* invariant under arbitrary rotations, we have proven the existence of inequivalent representations. Indeed,

$$\sigma(\pi(\widetilde{W}_0(\xi))) = \sigma(\pi(W(\xi, 0))) = \{e^{in\xi}\}_{n \in \mathbf{Z}} \quad (5.116)$$

since⁵ for $\psi_n : x \mapsto e^{inx}$

$$(\pi(W(\xi, 0))\psi_n)(x) = \psi_n(x + \xi) = e^{in\xi}\psi_n(x). \quad (5.117)$$

Even for irrational ξ , $\{e^{in\xi}\}_{n \in \mathbf{Z}}$, while dense in $S^1 \subset \mathbf{C}$, remains always strictly smaller. In any case, given a ξ , we can find a $\beta \in (0, 1)$ such that

$$\{e^{in\xi}\}_{n \in \mathbf{Z}} \not\equiv e^{i\beta\xi} \{e^{in\xi}\}_{n \in \mathbf{Z}}. \quad (5.118)$$

The example of a particle on S^1 introduces a new feature, a non trivial center of a C^* -algebra.

⁵Using the ansatz

$$(\pi(W(\xi, 0))\psi_n)(x) = \psi_n(x + \xi) = \alpha_n(\xi)\psi_n(x)$$

we find from multiple applications and the group property

$$\forall \xi \in \mathbf{R}, k \in \mathbf{Z} : (\alpha_n(\xi))^k = \alpha_n(k\xi) \wedge \alpha_n(2\pi) = 1$$

and therefore exponentials

$$\alpha_n(\xi) = e^{im\xi}$$

with $m \in \mathbf{Z}$. **WLOG** we can choose $m = n$.

Definition 5.19 (*commutant, center*). The *commutant* \mathcal{B}' of a subset \mathcal{B} of a C^* -algebra \mathcal{A} is the set of all elements of \mathcal{A} that commute with all elements of \mathcal{B} , i. e.

$$\mathcal{B}' = \{A \in \mathcal{A} : \forall B \in \mathcal{B} : AB = BA\}. \quad (5.119)$$

The *center* $Z(\mathcal{A})$ of a C^* -algebra \mathcal{A} is the set of all elements of \mathcal{A} that commute with all other elements, i. e.

$$Z(\mathcal{A}) = \mathcal{A}' \cap \mathcal{A}. \quad (5.120)$$

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Indeed, we can compute

$$\forall n, \nu \in \mathbf{Z} : V(2n\pi)U(\nu) = U(\nu)V(2n\pi)e^{i2n\nu\pi} = U(\nu)V(2n\pi) \quad (5.121)$$

and, of course,

$$V(2n\pi)V(\xi) = V(\xi)V(2n\pi) \quad (5.122)$$

or more generally

$$\forall n \in \mathbf{Z} : \widetilde{W}_0(2n\pi)\widetilde{W}_\nu(\xi) = \widetilde{W}_\nu(\xi)\widetilde{W}_0(2n\pi)e^{i2n\pi\nu} = \widetilde{W}_0(\xi)\widetilde{W}_\nu(2n\pi). \quad (5.123)$$

Therefore

$$\forall n \in \mathbf{Z} : \widetilde{W}_0(2n\pi) \in Z(\mathcal{A}_W^{S^1}) \quad (5.124)$$

but from

$$\widetilde{W}_0(2n\pi)\widetilde{W}_\nu(\xi) = e^{in\pi\nu}\widetilde{W}_\nu(\xi + 2n\pi) \quad (5.125)$$

and the fact that there is a priori *no* 2π -periodicity in the family of algebra elements $\widetilde{W}_\nu : \mathbf{R} \rightarrow \mathcal{A}_W^{S^1}$, we can expect that $\widetilde{W}_0(2n\pi) \neq \mathbf{1}$, i. e.

$$Z(\mathcal{A}_W^{S^1}) \neq \{z\mathbf{1} : z \in \mathbf{C}\}. \quad (5.126)$$

Indeed, computing the action of $\widetilde{W}_0(2n\pi)$ in the representations $(\mathcal{H}_{\alpha,\beta}, \pi_{\alpha,\beta})$, we find from (5.100)

$$(\pi_{\alpha,\beta}(\widetilde{W}_0(2n\pi))\psi)(x) = e^{i2n\pi\beta}\psi(x + 2n\pi) = e^{i2n\pi\beta}\psi(x), \quad (5.127)$$

i. e.

$$\pi_{\alpha,\beta}(\widetilde{W}_0(2n\pi)) = e^{i2n\pi\beta}\mathbf{1}. \quad (5.128)$$

For $0 < \beta < 1$, this is not compatible with $\widetilde{W}_0(2n\pi) = z\mathbf{1}$, because for all representations

$$\pi_{\alpha,\beta}(z\mathbf{1}) = z\pi_{\alpha,\beta}(\mathbf{1}) = z\mathbf{1}, \quad (5.129)$$

independently from β .

As mentioned in section 4.2 on page 68, symmetries are realized as automorphisms of the C^* -algebra of observables. For example the translations $x \mapsto x - \alpha$

$$\begin{aligned}\tau_\alpha : \mathcal{A}_W &\rightarrow \mathcal{A}_W \\ W(\xi, \eta) &\mapsto e^{-i\eta\alpha} W(\xi, \eta)\end{aligned}\quad (5.130)$$

and *Galileo boosts* $p \mapsto p + \beta$

$$\begin{aligned}\sigma_\beta : \mathcal{A}_W &\rightarrow \mathcal{A}_W \\ W(\xi, \eta) &\mapsto e^{i\xi\beta} W(\xi, \eta)\end{aligned}\quad (5.131)$$

form an abelian⁶ two-parameter group of automorphisms $\Gamma_{\alpha,\beta} : \mathcal{A}_W \rightarrow \mathcal{A}_W$, with $\alpha, \beta \in \mathbf{R}$,

$$\Gamma_{\alpha,\beta} = \sigma_\beta \circ \tau_\alpha = \tau_\alpha \circ \sigma_\beta \quad (5.132)$$

and⁷

$$\begin{aligned}\Gamma_{\alpha,\beta}(W(\xi, \eta)) &= \tau_\alpha(\sigma_\beta(W(\xi, \eta))) \\ &= e^{i(\xi\beta - \eta\alpha)} W(\xi, \eta) = W(-\alpha, -\beta) W(\xi, \eta) W(\alpha, \beta).\end{aligned}\quad (5.135)$$

As we have seen above (5.101), in a representation (\mathcal{H}, π) , the $\Gamma_{\alpha,\beta}$ are represented unitarily

$$\pi(\Gamma_{\alpha,\beta}(W(\xi, \eta))) = \pi(W(-\alpha, -\beta) W(\xi, \eta) W(\alpha, \beta)) = U_{\alpha,\beta}^* W(\xi, \eta) U_{\alpha,\beta} \quad (5.136)$$

by

$$U_{\alpha,\beta} = \pi(W(\alpha, \beta)). \quad (5.137)$$

However, while $\Gamma_{\alpha,\beta} : \mathcal{A}_W^{S^1} \rightarrow \mathcal{A}_W^{S^1}$ remains well defined on the circle for all $\alpha, \beta \in \mathbf{R}$, only $\Gamma_{\alpha,\nu}$ with $\alpha \in \mathbf{R}$ and $\nu \in \mathbf{Z}$ can be realized unitarily. We may say that we have a case of *Spontaneous Symmetry Breaking (SSB)*: the symmetry is an automorphism of the microscopic algebra of observables, but not of the representation corresponding to the macroscopic state. In

⁶Cf. section 3.8.4 on page 59 of [Ohl16].

⁷Note that

$$W(\alpha, 0)W(0, \beta) \neq W(0, \beta)W(\alpha, 0) \neq W(\alpha, \beta), \quad (5.133)$$

but

$$\begin{aligned}W(0, -\beta)(W(-\alpha, 0)AW(\alpha, 0))W(0, \beta) &= W(-\alpha, -\beta)AW(\alpha, \beta) \\ &= W(-\alpha, 0)(W(0, -\beta)AW(0, \beta))W(\alpha, 0),\end{aligned}\quad (5.134)$$

because the phases cancel.

the present case, all translations $\tau_\alpha = \Gamma_{\alpha,0}$ remain unbroken, but the Galileo boosts $\sigma_\nu = \Gamma_{0,\nu}$ are broken by the quantization of momenta, except for integer amounts.

In this example, the breaking of the symmetry is of course not fully spontaneous, because it is brought about by enforcing the non-trivial topology of the configuration space S^1 . Another way to describe the situation is that the classical boost symmetry $p \mapsto p + \beta$ is *anomalous* in the quantum theory or that we have an example of an *anomaly*, where a classical symmetry can not (fully) be realized in the quantum theory.

Nevertheless, there remains a weaker form of the Stone-von Neumann theorem, namely that all representations (\mathcal{H}, π) with $\pi(\widetilde{W}_0(2n\pi)) = \mathbf{1}$ are unitarily equivalent. In fact, we can show that there is a cyclic vector Ω_0 with

$$\forall \nu \in \mathbf{Z}, \xi \in \mathbf{R} : \left(\Omega_0, \pi_{\alpha,\beta}(\widetilde{W}_\nu(\xi))\Omega_0 \right) = \delta_{\nu,0} \quad (5.138)$$

corresponding to the state

$$\omega(\widetilde{W}_\nu(\xi)) = \delta_{\nu,0}. \quad (5.139)$$

Indeed from

$$\widetilde{W}_0(\xi)\widetilde{W}_0(\xi') = \widetilde{W}_0(\xi + \xi') \quad (5.140)$$

we infer

$$\widetilde{W}_0(\xi + 2\pi) = \widetilde{W}_0(2\pi)\widetilde{W}_0(\xi) \quad (5.141)$$

and

$$\pi(\widetilde{W}_0(\xi + 2\pi)) = \pi(\widetilde{W}_0(2\pi))\pi(\widetilde{W}_0(\xi)) = \pi(\widetilde{W}_0(\xi)). \quad (5.142)$$

Thus $\pi(\widetilde{W}_0(\xi))$ is periodic with period 2π and we are let to define

$$P_\pi = \int_0^{2\pi} \frac{d\xi}{2\pi} \pi(\widetilde{W}_0(\xi)). \quad (5.143)$$

Again, this is self-adjoint

$$\begin{aligned} P_\pi^* &= \int_0^{2\pi} \frac{d\xi}{2\pi} \pi(\widetilde{W}_0(-\xi)) = - \int_0^{-2\pi} \frac{d\xi}{2\pi} \pi(\widetilde{W}_0(\xi)) \\ &= \int_{-2\pi}^0 \frac{d\xi}{2\pi} \pi(\widetilde{W}_0(\xi)) = \int_0^{2\pi} \frac{d\xi}{2\pi} \pi(\widetilde{W}_0(\xi)) = P_\pi \end{aligned} \quad (5.144)$$

and $P_\pi \neq 0$ since

$$\pi(\widetilde{W}_{-\nu}(-\xi))P_\pi\pi(\widetilde{W}_\nu(\xi)) = \int_0^{2\pi} \frac{d\xi'}{2\pi} \pi(\widetilde{W}_{-\nu}(-\xi)\widetilde{W}_0(\xi')\widetilde{W}_\nu(\xi))$$

$$= \int_0^{2\pi} \frac{d\xi'}{2\pi} e^{i\nu\xi'} \pi(\widetilde{W}_0(\xi')) \quad (5.145)$$

can not vanish, because it's the ν th Fourier coefficient of the periodic unitary operator $\pi(\widetilde{W}_0(\xi))$. Finally, similarly to the proof of the Stone-von Neumann

$$P_\pi \pi(\widetilde{W}_\nu(\xi)) P_\pi = \delta_{\nu,0} P_\pi. \quad (5.146)$$

Thus we see that P_π is an orthogonal projection and can construct Ω_0 as a normalized eigenvector of P_π .

5.4 Reducible Representations

A representation (\mathcal{H}, π) of a C^* -algebra \mathcal{A} corresponding to an impure state

$$\omega = \sum_{i=1}^N p_i \omega_i \quad (5.147)$$

with each ω_i pure, $\mathbf{R} \ni p_i > 0$ and $\sum_{i=1}^N p_i = 1$, will not be reducible and can be written as a direct sum (or integral) of N irreducible representations

$$\mathcal{H} = \bigoplus_{i=1}^N \mathcal{H}_i \quad (5.148a)$$

$$\pi = \bigoplus_{i=1}^N \pi_i, \quad (5.148b)$$

where we must allow that not all (\mathcal{H}_i, π_i) are different. Since all (\mathcal{H}_i, π_i) in (5.148) are, by definition, irreducible, they correspond to the pure states

$$\begin{aligned} \omega_i : \mathcal{A} &\rightarrow \mathbf{C} \\ A &\mapsto \omega_i(A) = (\Omega_i, \pi_i(A)\Omega_i) \end{aligned} \quad (5.149)$$

with $\Omega_i \in \mathcal{H}_i$. Therefore the state ω has the decomposition

$$\begin{aligned} \omega : \mathcal{A} &\rightarrow \mathbf{C} \\ A &\mapsto \omega(A) = \sum_{i=1}^N p_i (\Omega_i, \pi_i(A)\Omega_i). \end{aligned} \quad (5.150)$$

Introducing the operator

$$\begin{aligned} \rho : \mathcal{H} &\rightarrow \mathcal{H} \\ \Psi &\mapsto \sum_{i=1}^N \Omega_i (\Omega_i, \Psi) p_i, \end{aligned} \quad (5.151)$$

we can express the state ω as a trace

$$\omega(A) = \text{tr}_{\mathcal{H}}(\rho\pi(A)). \quad (5.152)$$

This suggests the pair of definitions

Definition 5.20. A bounded operator $A \in \mathcal{L}(\mathcal{H})$ in a separable Hilbert space \mathcal{H} is called *trace class*, **iff** the sum (or integral) for any orthonormal basis $\{\psi_i\}_{i \in I \subseteq \mathbf{N}}$ satisfies

$$\text{tr}_{\mathcal{H}}(|A|) = \text{tr}_{\mathcal{H}}(\sqrt{A^*A}) = \sum_{i \in I} (\psi_i, \sqrt{A^*A}\psi_i) < \infty. \quad (5.153)$$

Then the *trace* of A

$$\text{tr}_{\mathcal{H}}(A) = \sum_{i \in I} (\psi_i, A\psi_i) \quad (5.154)$$

is absolutely convergent and *independent* of the basis.

Definition 5.21. A positive trace class operator $\rho \in \mathcal{L}(\mathcal{H})$ with

$$\text{tr}_{\mathcal{H}}(\rho) = 1 \quad (5.155)$$

is called a *density matrix*.

Theorem 5.22. Given a representation (\mathcal{H}, π) of a C^* -algebra \mathcal{A} , each density matrix $\rho : \mathcal{H} \rightarrow \mathcal{H}$ defines a state

$$\begin{aligned} \omega_{\rho} : \mathcal{A} &\rightarrow \mathbf{C} \\ A &\mapsto \text{tr}_{\mathcal{H}}(\rho\pi(A)). \end{aligned} \quad (5.156)$$

Proof. ω_{ρ} is normalized by definition

$$\omega_{\rho}(\mathbf{1}) = \text{tr}_{\mathcal{H}}\rho = 1 \quad (5.157)$$

and since positivity implies self-adjointness, we can diagonalize ρ

$$\rho : \phi \mapsto \sum_{i \in I} p_i \psi_i (\psi_i, \phi) \quad (5.158)$$

with $\mathbf{R} \ni p_i > 0$ and $\sum_{i \in I} p_i = 1$. Thus

$$\begin{aligned} \omega_{\rho}(A^*A) &= \text{tr}_{\mathcal{H}}(\rho\pi(A^*A)) = \sum_{i \in I} (\psi_i, \rho\pi(A^*A)\psi_i) \\ &= \sum_{i, j \in I} p_j \underbrace{(\psi_i, \psi_j)}_{=\delta_{ij}} (\psi_j, \pi(A^*A)\psi_i) = \sum_{i \in I} p_j (\psi_i, \pi(A^*A)\psi_i) \\ &= \sum_{i \in I} \underbrace{p_j}_{\geq 0} \underbrace{(\pi(A)\psi_i, \pi(A)\psi_i)}_{\geq 0} \geq 0. \end{aligned} \quad (5.159)$$

□

Remark 5.23. The pure states correspond to density matrices that are one-dimensional projections

$$\rho : \Phi \mapsto \Psi_\rho (\Psi_\rho, \Phi) \quad (5.160)$$

with $\|\Psi_\rho\| = 1$.

Proof.

$$\begin{aligned} \omega_\rho(A) &= \sum_{i \in I} (\psi_i, \rho \pi(A) \psi_i) = \sum_{i \in I} (\psi_i, \Psi_\rho) (\Psi_\rho, \pi(A) \psi_i) \\ &= \sum_{i \in I} (\Psi_\rho, \pi(A) \psi_i) (\psi_i, \Psi_\rho) = (\Psi_\rho, \pi(A) \Psi_\rho), \end{aligned} \quad (5.161)$$

using the completeness of the basis $\{\psi_i\}_{i \in I \subseteq \mathbf{N}}$:

$$\forall \phi \in \mathcal{H} : \sum_{i \in I} \psi_i (\psi_i, \phi) = \phi. \quad (5.162)$$

□

Note that density matrices appear here in two very different ways. First we have decomposed a reducible representation and written the state as a sum of pure vector states

$$\omega : A \mapsto \sum_{i=1}^N p_i (\Omega_i, \pi_i(A) \Omega_i), \quad (5.163)$$

which we have rewritten as a trace of a density matrix in the big Hilbert space. But after introducing the notion of a density matrix, we have noticed that *every* density matrix defines a state

$$\omega_\rho : A \mapsto \text{tr}_{\mathcal{H}}(\rho \pi(A)), \quad (5.164)$$

which can be pure or not.

Lecture 20: Fri, 21. 12. 2018

Example 5.24. Consider the C^* -algebra \mathcal{M}_2 of 2×2 -Matrices, for which furnish their own representation $(\mathcal{H}, \pi) = (\mathbf{C}^2, \text{id})$. As we have also seen in the exercises, the states on \mathcal{M}_2 can be parametrized by three real numbers $\vec{\alpha}$ with $|\vec{\alpha}| \leq 1$ and

$$\begin{aligned} \omega_{\vec{\alpha}} : \mathcal{M}_2 &\rightarrow \mathbf{C} \\ M &\mapsto \text{tr}(M \rho(\vec{\alpha})), \end{aligned} \quad (5.165)$$

where

$$\rho(\vec{\alpha}) = \frac{1}{2} (\mathbf{1} + \vec{\alpha}\vec{\sigma}) . \quad (5.166)$$

The pure states are those with $|\vec{\alpha}| = 1$. It is important to realize that the representation $(\mathbf{C}^2, \text{id})$ is in general *not* the GNS-representation associated to the state $\omega_{\vec{\alpha}}$ or even unitarily equivalent to it.

In fact, the dimension of the representation space $\mathcal{H}_{\vec{\alpha}}$ of the GNS-representation $(\mathcal{H}_{\vec{\alpha}}, \pi_{\vec{\alpha}})$ constructed from $\omega_{\vec{\alpha}}$ depends on the dimension of the Gel'fand ideal

$$\mathcal{I}_{\vec{\alpha}} = \{M \in \mathcal{M}_2 : \omega_{\vec{\alpha}}(M^*M) = 0\} \quad (5.167)$$

via

$$\dim \mathcal{H}_{\vec{\alpha}} = \dim (\mathcal{M}_2 / \mathcal{I}_{\vec{\alpha}}) = \underbrace{\dim \mathcal{M}_2}_{=4} - \dim \mathcal{I}_{\vec{\alpha}} . \quad (5.168)$$

Iff $\omega_{\vec{\alpha}}$ is not a pure state, $\rho(\vec{\alpha})$ has rank 2, the Gel'fand ideal is trivial and $\dim \mathcal{H}_{\vec{\alpha}} = 4$. **OTOH**, iff $\omega_{\vec{\alpha}}$ is pure, $\rho(\vec{\alpha})$ is a projector with rank 1, the Gel'fand ideal has dimension 2 (all matrices with vanishing first column) and $\dim \mathcal{H}_{\vec{\alpha}} = 2$.

5.4.1 Fell's Theorem

In order to organize the representations, we make the following

Definition 5.25 (*vector states*). Given a state ω over a C^* -algebra \mathcal{A} and the corresponding representation $(\mathcal{H}_{\omega}, \pi_{\omega})$, we call the members of the set

$$\{\omega_{\Psi} : \Psi \in \mathcal{H}_{\omega}, \|\Psi\| = 1\} \quad (5.169)$$

with

$$\begin{aligned} \omega_{\Psi} : \mathcal{A} &\rightarrow \mathbf{C} \\ A &\mapsto (\Psi, \pi_{\omega}(A)\Psi) , \end{aligned} \quad (5.170)$$

the *vector states* of this representation.

Definition 5.26 (*folium*). Given a state ω over a C^* -algebra \mathcal{A} and the corresponding representation $(\mathcal{H}_{\omega}, \pi_{\omega})$, we call set

$$\{\omega_{\rho} : \rho \in \mathcal{L}(\mathcal{H}_{\omega}), 0 \leq \rho \text{ trace class, } \text{tr}_{\mathcal{H}_{\omega}} \rho = 1\} \quad (5.171)$$

with

$$\begin{aligned} \omega_{\rho} : \mathcal{A} &\rightarrow \mathbf{C} \\ A &\mapsto \text{tr}_{\mathcal{H}_{\omega}}(\rho \pi_{\omega}(A)) , \end{aligned} \quad (5.172)$$

the *folium* of this representation.

Besides the norm topology on the space \mathcal{A}^* of all linear functionals $\omega : \mathcal{A} \rightarrow \mathbf{C}$,

$$\|\omega\| = \sup_{A \in \mathcal{A}, \|A\|=1} |\omega(A)|, \quad (4.1)$$

introduced earlier, we can define a weaker topology, that is probably physically better motivated. We start from a family of seminorms

$$\begin{aligned} \|\cdot\|_{A_1, \dots, A_n} : \mathcal{A}^* &\rightarrow \mathbf{R}_+ \cup \{0\} \\ \omega &\mapsto \sup_{k=1, \dots, n} |\omega(A_k)| \end{aligned} \quad (5.173)$$

indexed by finite subsets of \mathcal{A} with $\|A_k\| = 1$ and define neighborhoods of the origin

$$\begin{aligned} N(\epsilon; A_1, \dots, A_n) &= \{\omega \in \mathcal{A}^* : \|\omega\|_{A_1, \dots, A_n} < \epsilon\} \\ &= \{\omega \in \mathcal{A}^* : \forall k = 1, \dots, n : |\omega(A_k)| < \epsilon\}. \end{aligned} \quad (5.174)$$

The set of all such neighborhoods define the *weak topology* on \mathcal{A}^* . Because we use only a *finite* number of elements of \mathcal{A} to measure distances, the neighborhoods in the weak topology are cylinders, while the neighborhoods in the norm topology are balls. Therefore, the open sets in the weak topology can be generated by *infinite unions* of open sets in the norm topology, but the open sets in the norm topology can not be generated as *finite intersections* of open sets in the weak topology. Thus every open set in the weak topology is also an open set in the norm topology, but not vice-versa (unless \mathcal{A} is finite-dimensional). Therefore, the weak topology is coarser than the norm topology.

Since only a finite number of measurements can be performed in real life, the weak topology appears to be more appropriate than the norm topology, which can only be realized when *all* observables are measured.

The importance of the notion of folium is explained by

Theorem 5.27 (Fell [Fel60]). *The folium of a faithful representation of a C^* -algebra \mathcal{A} is weakly dense in the set of all states on \mathcal{A} .*

This theorem is however somewhat disappointing, because it means that one can not distinguish folia by physically realizable measurements, which necessarily involve only a finite number of observables. In particular, if we have two states ω and ω' corresponding to unitarily inequivalent representations, states from one folium can nevertheless be approximated arbitrarily well by states from the other folium. Thus one might conclude that “one folium fits all” or “one Hilbert space fits them all”.

This conclusion is wrong, however, because the qualitative description of important physical phenomena like *SSB* and *phase transitions* depend on idealizations like the *infinite volume limit*, which corresponds to choosing values for an infinite number of measurements. While a sample of finite size will never undergo a phase transition or *SSB*, we get a better description of its properties by making this idealization.

—6—

INFINITE NUMBER OF DEGREES OF FREEDOM

6.1 Fock Representation

Coming back to example 5.8, i. e. the Weyl algebra \mathcal{A}_W for quantum mechanics with n d.o.f.

$$W(\xi_1, \eta_1)W(\xi_2, \eta_2) = e^{\frac{i}{2} \sum_{i=1}^n (\xi_{1,i}\eta_{2,i} - \eta_{1,i}\xi_{2,i})} W(\xi_1 + \xi_2, \eta_1 + \eta_2) \quad (6.1)$$

we will now allow for $n \rightarrow \infty$. Of course, in order to have a well defined multiplication law (6.1), we must assume that $\xi, \eta \in l_2$.

Since we know from Stone's theorem that

$$\pi(x_i) = -i \lim_{\eta \rightarrow 0} \frac{\partial}{\partial \eta_i} \pi(W(0, \eta)) \quad (6.2a)$$

$$\pi(p_i) = -i \lim_{\xi \rightarrow 0} \frac{\partial}{\partial \xi_i} \pi(W(\xi, 0)) \quad (6.2b)$$

are well defined symmetric operators on a common dense domain in any regular representation (\mathcal{H}, π) , we can suppress the π and write

$$W(\xi, \eta) = \exp \left(i \sum_{i=1}^n (\xi_i p_i + \eta_i x_i) \right). \quad (6.3)$$

For convenience, we will in the following also use representations of the Heisenberg algebra \mathcal{A}_H , generated by the symmetric x_i, p_i with

$$[x_i, p_j] = i\delta_{ij} \quad (6.4a)$$

$$[x_i, x_j] = [p_i, p_j] = 0. \quad (6.4b)$$

Since x_i and p_i can not be simultaneously bounded, \mathcal{A}_H is *not* a C^* -algebra and we have to specify a representation (\mathcal{H}, π) and a common dense domain $D \subset \mathcal{H}$. Nevertheless we will in the following often write x_i for $\pi(x_i)$ and p_i for $\pi(p_i)$ and leave the representation implicit.

As suggested by the proof of the Stone-von Neumann theorem 5.12 and also discussed in the exercises, one can introduce

$$a_i = \frac{1}{\sqrt{2}}(x_i + ip_i) \quad (6.5a)$$

$$a_i^* = \frac{1}{\sqrt{2}}(x_i - ip_i), \quad (6.5b)$$

the (unbounded) *annihilation and creation operators*, together with the *number operators*

$$N_i = a_i^* a_i \quad (6.5c)$$

for all $i \in \mathbf{N}$. Obviously, one has

$$[a_i, a_j^*] = \delta_{ij} \quad (6.6a)$$

$$[a_i, a_j] = [a_i^*, a_j^*] = 0 \quad (6.6b)$$

and

$$[N_i, a_j] = -\delta_{ij} a_i \quad (6.6c)$$

$$[N_i, a_j^*] = \delta_{ij} a_i^*. \quad (6.6d)$$

Theorem 6.1. *In an irreducible representation (\mathcal{H}, π) of the Heisenberg algebra \mathcal{A}_H with domain $D \subset \mathcal{H}$, the following are equivalent (suppressing the π):*

1. *The total number operator*

$$N = \sum_{i=1}^{\infty} N_i \quad (6.7)$$

exists in the sense that the strong limit

$$\forall \alpha \in \mathbf{R} : \text{s-lim}_{n \rightarrow \infty} e^{i\alpha \sum_{i=1}^n a_i^* a_i} = e^{i\alpha N} = T(\alpha) \quad (6.8)$$

exists on D and defines a one-parameter group of unitary operators $T(\alpha) : D \rightarrow D$, that is strongly continuous in α , such that N exists as its generator.

2. *There exists a non-zero vector $\Omega \in \mathcal{H}$, called the Fock state, or Fock vacuum state, such that*

$$\forall i : a_i \Omega = 0. \quad (6.9)$$

Such a representation is called a Fock representation $(\mathcal{H}, \pi, \Omega)$.

Proof. From 1) and the commutation relations, we see, using the Hausdorff formula

$$e^A B e^{-A} = e^{[A, \cdot]} B \quad (6.10)$$

that

$$T(\alpha) a_i (T(\alpha))^{-1} = e^{-i\alpha} a_i. \quad (6.11)$$

Therefore

$$T(2\pi) a_i = a_i T(2\pi) \quad (6.12)$$

or

$$[T(2\pi), a_i] = 0. \quad (6.13)$$

Analogously

$$[T(2\pi), a_i^*] = 0, \quad (6.14)$$

i. e.

$$[T(2\pi), \mathcal{A}_H] = 0. \quad (6.15)$$

Since the representation is irreducible, this implies that $T(2\pi)$ is proportional to $\mathbf{1}_{\mathcal{H}}$ and since it is unitary

$$\exists \theta \in \mathbf{R} : T(2\pi) = e^{i\theta} \mathbf{1}_{\mathcal{H}}. \quad (6.16)$$

This suggests to define

$$T'(\alpha) = e^{-i\alpha\theta/2\pi} T(\alpha) \quad (6.17)$$

with

$$T'(2\pi) = \mathbf{1}_{\mathcal{H}}. \quad (6.18)$$

In the spectral decomposition

$$T'(\alpha) = \int_{\sigma(N')} dP(\lambda) e^{i\alpha\lambda}, \quad (6.19)$$

where

$$N' = N - \frac{\theta}{2\pi}, \quad (6.20)$$

this means that

$$\forall \lambda \in \sigma(N') : e^{i2\pi\lambda} = 1. \quad (6.21)$$

Thus $\sigma(N') \subseteq \mathbf{Z}$ and $\sigma(N)$ is also discrete. If $\sigma(N) \ni \lambda > 0$ with Ψ_λ one of the corresponding eigenvectors¹, we have

$$0 < \lambda \|\Psi_\lambda\|^2 = (\Psi_\lambda, N\Psi_\lambda) = \sum_i \|a_i \Psi_\lambda\|^2 \quad (6.22)$$

¹ λ might be a degenerate eigenvalue.

and therefore

$$\exists i \in \mathbf{N} : a_i \Psi_\lambda \neq 0. \quad (6.23)$$

Using the commutation relations

$$T(\alpha) a_i \Psi_\lambda = a_i e^{-i\alpha} T(\alpha) \Psi_\lambda = a_i e^{-i\alpha} e^{i\alpha N} \Psi_\lambda = a_i e^{-i\alpha} e^{i\alpha \lambda} \Psi_\lambda = e^{i(\lambda-1)\alpha} a_i \Psi_\lambda \quad (6.24)$$

we conclude that a_i acts as a lowering operator and $\lambda - 1 \in \sigma(N)$ as well. **OTOH** $N \geq 0$, because it is a sum of positive operators and therefore $\sigma(N) \subseteq \mathbf{R}_+ \cup \{0\}$. Thus there must be a $\lambda \in \sigma(N)$ where the lowering terminates for all a_i . This λ must be 0, i. e.

$$\forall i : a_i \Psi_0 = 0, \quad (6.25)$$

and we have shown 2) with $\Omega = \Psi_0$.

The other direction follows from

$$\mathcal{A}_H \Omega = \mathcal{P}(a^*) \Omega \subset \mathcal{H}, \quad (6.26)$$

where $\mathcal{P}(a^*)$ are the polynomials in $\{a_i^*\}$. $\mathcal{P}(a^*) \Omega$ is dense in \mathcal{H} , since the representation is irreducible. N exists on $\mathcal{P}(a^*) \Omega$ by construction and the exponential series for $T(\alpha)$ converges strongly to a one parameter group of unitary operators. \square

Remark 6.2. The eigenvalue 0 of the total number operator N is not degenerate in a Fock representation $(\mathcal{H}, \pi, \Omega)$, i. e. Ω is unique.

Proof. Assume that there is a second $\mathcal{H} \ni \Omega' \neq \Omega$ with $N\Omega' = 0$. Then, since $N \geq 0$,

$$\forall i : a_i \Omega' = 0 \quad (6.27)$$

and for every polynomial P

$$(\Omega', P(a^*) \Omega) = (P(a) \Omega', \Omega) = 0 \quad (6.28)$$

and $\Omega' = 0$, because Ω is cyclic. \square

Lecture 21: Wed, 09.01.2019

We can rescue a part of the Stone-von Neumann to the case of an infinite number of **d.o.f.**

Theorem 6.3. *All irreducible Fock representations $(\mathcal{H}, \pi, \Omega)$ of a Heisenberg algebra \mathcal{A}_H are unitarily equivalent.*

Therefore, it makes sense to speak of “the” Fock representation.

Proof. Given two irreducible Fock representations $\{(\mathcal{H}_i, \pi_i, \Omega_i)\}_{i=1,2}$ of \mathcal{A}_H , we can use the cyclic vectors Ω_1 and Ω_2 to define a map $U : \mathcal{H}_1 \supseteq \pi_1(\mathcal{A}_H)\Omega_1 \rightarrow \pi_2(\mathcal{A}_H)\Omega_2 \subseteq \mathcal{H}_2$ and its inverse $U^{-1} : \pi_2(\mathcal{A}_H)\Omega_2 \rightarrow \pi_1(\mathcal{A}_H)\Omega_1$ on dense subsets by

$$\forall A \in \mathcal{A}_H : U\pi_1(A)\Omega_1 = \pi_2(A)\Omega_2 \quad (6.29a)$$

$$\forall A \in \mathcal{A}_H : U^{-1}\pi_2(A)\Omega_2 = \pi_1(A)\Omega_1, \quad (6.29b)$$

including $U\Omega_1 = \Omega_2$ and $U^{-1}\Omega_2 = \Omega_1$. Since we can compute the matrix elements

$$(\pi_i(A)\Omega_i, \pi_i(B)\Omega_i) \quad (6.30)$$

for all $A, B \in \mathcal{P}(a^*)$ from the commutation relations and the condition $a_j\Omega_i = 0$, they can't depend on the representation and U must be unitary. \square

The polynomials in the creation operators $\{\pi(a_i^*)\}$ applied to Ω form a dense subset of \mathcal{H} , in which all vectors are simultaneous eigenvectors of all the $\{N_i\}$. Therefore the Fock representation is also called the *occupation number representation*.

Note that we have not yet made any reference to the dynamics of the system. The Fock representation exists independently of the Hamiltonian. In the case of decoupled harmonic oscillators it provides a diagonalization of the Hamiltonian

$$H_0 = \sum_i \omega_i a_i^* a_i + \text{const.}, \quad (6.31)$$

but it exists also for free particles and other potentials. However *only* in the case (6.31) is Ω the ground state of the system. By the Stone-von Neumann theorem, the Fock representation is as good as any other representation for a finite number of **d.o.f.**, even though there might be computationally more convenient choices.

In the case of an infinite number of **d.o.f.**, this is no longer true. As we shall see below, one can show with simple arguments that interacting relativistic quantum fields and many body systems with non-zero density can *not* be described by a Fock representation or the folium of a Fock representation.

The Fock representation a good representation, **iff** the total occupation number N is a good quantum number for the description of the system. Assume that the system described by (6.31) has a *mass gap* or *energy gap* m

$$\forall i : \omega_i \geq m > 0. \quad (6.32)$$

Then the series defining H_0 dominates the series defining N term by term and

$$N \leq \frac{1}{m} H_0 \quad (6.33)$$

and the existence of H_0 as a self adjoint operator implies the existence of the number operator N .

6.2 Non-Fock Representations

Consider a many particle system with non-zero density n in the thermodynamic limit

$$\left. \begin{array}{l} N \rightarrow \infty \\ V \rightarrow \infty \end{array} \right\} \quad n = \frac{N}{V} = \text{const.} > 0. \quad (6.34)$$

If N_V denotes the number operator in the volume N_V , we have

$$N_V \leq N \quad (6.35)$$

and therefore

$$\forall \Psi \in D(N) : \|n\Psi\| = \lim_{V \rightarrow \infty} \frac{1}{V} \|N_V \Psi\| \leq \lim_{V \rightarrow \infty} \frac{1}{V} \underbrace{\|N\Psi\|}_{< \infty} = 0. \quad (6.36)$$

Thus if $n \neq 0$, the total number operator N *must not* be a well defined operator with a dense domain. Thus we cannot use the Fock representation to describe such a system.

6.2.1 Haag's Theorem

A typical description of a physical system starts from a Hamiltonian for “free” or “non-interacting” states that can be described by decoupled harmonic oscillators (6.31) in the bosonic case (e.g. photons, phonons and Cooper pairs) and by anti-commuting annihilation and creation operators in the fermionic case (e.g. electrons). These states can be simple plane waves for photons in the vacuum or or complicated wave functions for electrons in condensed matter.

The interactions among these states is then taken into account by adding a polynomial in the annihilation and creation operators

$$H_g = \underbrace{\sum_i \omega_i a_i^* a_i}_{= H_0} + g H_{\text{int.}}(a, a^*). \quad (6.37)$$

If we could diagonalize the Hamiltonian (6.2.1) explicitly, we could introduce annihilation and creation operators A_i and A_i^* for the interacting modes with energies E_i and write the Hamiltonian in a quadratic form

$$H_g = \sum_i E_i A_i^* A_i + \text{const.} . \quad (6.38)$$

A more realistic way to solve the dynamics of (6.2.1) is to first solve the linear Heisenberg picture equations corresponding to H_0

$$\frac{d}{dt} a_i^{(0)}(t) = i[H_0, a_i^{(0)}(t)] = -i\omega_i a_i^{(0)}(t) \quad (6.39)$$

as

$$a_i^{(0)}(t) = e^{iH_0 t} a_i(0) e^{-iH_0 t} = e^{-i\omega_i t} a_i(0) . \quad (6.40)$$

OTOH, the full Heisenberg equations of motion

$$\frac{d}{dt} a_i(t) = i[H_g, a_i(t)] \quad (6.41)$$

for $g \neq 0$ are not linear and we can not derive a closed expression for

$$a_i(t) = e^{iHt} a_i(0) e^{-iHt} . \quad (6.42)$$

The compatibility of matrix elements in the Heisenberg and Schrödinger picture

$$\begin{aligned} (\Psi_H, a_H(t) \Phi_H) &= (\Psi, e^{iH_g t} a e^{-iH_g t} \Phi) \\ &= (e^{-iH_g t} \Psi, a e^{-iH_g t} \Phi) = (\Psi_S(t), a_S \Phi_S(t)) \end{aligned} \quad (6.43)$$

can be extended to the interaction picture

$$\begin{aligned} (\Psi_H, a_H(t) \Phi_H) &= (e^{iH_0 t} e^{-iH_g t} \Psi, e^{iH_0 t} a e^{-iH_0 t} e^{iH_0 t} e^{-iH_g t} \Phi) \\ &= (U(t) \Psi, a^{(0)}(t) U(t) \Phi) = (\Psi, U^*(t) a^{(0)}(t) U(t) \Phi) \end{aligned} \quad (6.44)$$

where

$$a^{(0)}(t) = e^{iH_0 t} a e^{-iH_0 t} \quad (6.45a)$$

$$U(t) = e^{iH_0 t} e^{-iHt} = 1 + \mathcal{O}(g) . \quad (6.45b)$$

Since

$$i \frac{d}{dt} U(t) = e^{iH_0 t} (H_g - H_0) e^{-iH_g t} = g e^{iH_0 t} H_{\text{int.}} e^{-iH_g t}$$

$$= g e^{iH_0 t} H_{\text{int.}} e^{-iH_0 t} e^{iH_0 t} e^{-iH_g t} = g H_{\text{int.}}(t) U(t), \quad (6.46)$$

the *formally* unitary operator $U(t) = (U^*(t))^{-1}$ satisfies

$$i \frac{d}{dt} U(t) = g H_{\text{int.}}(t) U(t) \quad (6.47a)$$

with a *time dependent* interaction

$$H_{\text{int.}}(t) = e^{iH_0 t} H_{\text{int.}}(a, a^*) e^{-iH_0 t} = H_{\text{int.}}(a^{(0)}(t), (a^{(0)}(t))^*). \quad (6.47b)$$

The Schrödinger equation (6.47a) is solved formally by *Dyson's series*

$$U(t) = \mathbb{T} e^{-ig \int_0^t dt' H_{\text{int.}}(t')}. \quad (6.48)$$

Matrix elements involving Dyson's series (6.48) are evaluated in perturbation theory using *Feynman diagrams*, for example,

$$g H_{\text{int.}}(a, a^*) = g \sum_{klm} v_{klm} a_k^* a_l^* a_m + \text{h. c.} \equiv -ig \begin{array}{c} a_k^* \quad a_l^* \\ \searrow \quad \nearrow \\ \bullet \quad v_{klm} \\ \uparrow \\ a_m \end{array}. \quad (6.49)$$

A typical second order contribution is then

$$-g^2 \begin{array}{c} a_k^* \quad a_l^* \\ \searrow \quad \nearrow \\ \bullet \quad \sum_j v_{kjm} v_{ljn} \\ \nearrow \quad \searrow \\ a_m \quad a_n \end{array}. \quad (6.50)$$

While perturbation theory has been spectacularly successful for predicting scattering cross sections in weakly interacting theories, it can fail spectacularly if the couplings become large or if the energy denominators appearing in the perturbation series become small.

This brings us to the question under which circumstances the operator $U(t)$ in (6.47a) is a well defined unitary operator relating the free and interacting modes. The surprising and somewhat disappointing answer is provided by

Theorem 6.4 (Haag (1955)). *Let \mathcal{A}_H be the Heisenberg algebra of a system with local degrees of freedom that is invariant under translations. Let $(\mathcal{H}_0, \pi_0, \Omega_0)$ and $(\mathcal{H}_g, \pi_g, \Omega_g)$ be the Fock representations of \mathcal{A}_H in which the free and interacting Hamiltonians H_0 and H_g , respectively, are well defined. These representations are not unitarily equivalent for $g \neq 0$.*

Thus the interaction picture does *not* exist for translation invariant systems, such as quantum field theories in the vacuum.

Before we can prove this theorem, we must define the systems under consideration more precisely. By “local degrees of freedom”, we mean that the annihilation and creation operators spanning the corresponding \mathcal{A}_H can be written in terms of operator valued distributions

$$a_i = \psi(f_i) = \int d^n x \bar{f}_i(x) \psi(x) \quad (6.51)$$

with

$$[\psi(x), \psi^*(x')] = \delta^n(x - x') \quad (6.52)$$

and $\{f_i\}$ an orthonormal set of square integrable smooth functions. Conversely, if $\{f_i\}$ is complete

$$\psi(x) = \sum_i f_i(x) a_i. \quad (6.53)$$

Note that x corresponds to a point in space, *not* an event in space time. The dynamics is given by the temporal evolution of the a_i or $\psi(x)$. If we have defined the annihilation and creation operators as in (6.51), we can implement space translations straightforwardly

$$\begin{aligned} \alpha : \mathbf{R}^n \times \mathcal{A}_H &\rightarrow \mathcal{A}_H \\ (\xi, \psi(f)) &\mapsto \alpha_\xi(\psi(f)) = \psi(\tilde{\alpha}_\xi f) \end{aligned} \quad (6.54a)$$

where

$$(\tilde{\alpha}_\xi f)(x) = f(x - \xi). \quad (6.54b)$$

One could also write formally

$$(\alpha_\xi \psi)(x) = \psi(x + \xi), \quad (6.55)$$

but this is only to be understood as a short hand, because the ψ are distributions and $\psi(x)$ is *not* an element of \mathcal{A}_H .

Obviously, each α_ξ defines a *-automorphism of \mathcal{A}_H and α therefore a n -parameter family of *-automorphisms.

Lecture 22: Fri, 11. 01. 2019

Lemma 6.5. *In an irreducible Fock representation $(\mathcal{H}, \pi, \Omega)$ of a \mathcal{A}_H that is invariant under translations (6.54), the Fock state Ω is the unique translation invariant state and space translations are implemented by strongly continuous unitary operators $U(\xi)$, i. e.*

$$U(\xi)\psi(x)U^*(\xi) = \psi(x + \xi). \quad (6.56)$$

Proof. In a Fock representation, the number operator

$$\begin{aligned} N &= \sum_i a_i^* a_i = \sum_i \int d^n x d^n y f_i(x) \bar{f}_i(y) \psi^*(x) \psi(y) \\ &= \int d^n x d^n y \delta^n(x-y) \psi^*(x) \psi(y) = \int d^n x \psi^*(x) \psi(x) \end{aligned} \quad (6.57)$$

is well defined. It is obviously invariant under translations $\alpha_\xi(N) = N$ and the spacial translations commute with it $[U(\xi), N] = 0$. Thus it suffices to study the sectors of \mathcal{H} with a fixed number of modes, **WLOG** m . These correspond to wavefunctions for m “particles”, i.e. square integrable functions $\mathbf{R}^{nm} \rightarrow \mathbf{C}$. But square integrable functions must fall off in every direction for $nm > 0$ and can not be invariant under the action of $\tilde{\alpha}_\xi$, i.e. constant shift $\forall i = 1, \dots, m : x_i \mapsto x_i + \xi$. Thus only the 0-“particle” state Ω is translation invariant. \square

Proof of Haag’s Theorem 6.4. By lemma 6.5, Ω_g is the unique translation invariant state with $U_g(\xi)\Omega_g = \Omega_g$ for all members of the family of translation operators $U_g(\xi)$. **OTOH**, assume that $(\mathcal{H}_0, \pi_0, \Omega_0)$ is another, unitarily equivalent Fock representation for a_i and a_i^* . Then we can use *both* $U_g(\xi)$ and $U_0(\xi)$ to translate them

$$U_g(\xi)a_i U_g^*(\xi) = \alpha_\xi(a_i) \quad (6.58a)$$

$$U_0(\xi)a_i U_0^*(\xi) = \alpha_\xi(a_i) \quad (6.58b)$$

and find

$$U_g(\xi)U_0^*(\xi)a_i U_0(\xi)U_g^*(\xi) = \alpha_\xi(\alpha_{-\xi}(a_i)) = a_i. \quad (6.59)$$

Therefore

$$[U_g U_0^{-1}, a_i] = [U_g U_0^{-1}, a_i^*] = 0 \quad (6.60)$$

or

$$[U_g U_0^{-1}, \pi(\mathcal{A}_H)] = 0 \quad (6.61)$$

for all representations π . In an irreducible representation, we must have

$$U_g(\xi)U_0^{-1}(\xi) = \mathbf{1}e^{i\theta_g(\xi)} = \mathbf{1}e^{i\theta_g \cdot \xi}, \quad (6.62)$$

where the second equation is a result of the group property. We can therefore absorb the phase $e^{i\theta_g \cdot \xi}$

$$U_g(\xi) \rightarrow U_g(\xi)e^{-i\theta_g \cdot \xi} \quad (6.63)$$

to get

$$U_g(\xi) = U_0(\xi). \quad (6.64)$$

Finally, if the translation operators are the same, the unique states left invariant by them must also be the same

$$\Omega_g = \Omega_0. \quad (6.65)$$

This implies that the Fock state, which is the lowest energy state in both representations agrees, which can not be, since the interaction changes the ground state. \square

In fact, this phenomenon can be demonstrated already by very simple examples

Example 6.6. While this caricature

$$H = \sum_i \omega_i a_i^* a_i + g \sum_i (\bar{j}_i a_i + j_i a_i^*) \quad (6.66)$$

of a scalar field $\psi \propto a + a^*$ coupled to a classical “source” $\{j_i\}$ can be diagonalized exactly for any $\{j_i\}$ and the corresponding unitary operators intertwining between $g = 0$ and $g \neq 0$ can be constructed explicitly for each mode *separately*, the product exists only if $\sum_i |j_i|^2 / \omega_i^2 \leq \infty$. In a local theory, we will have $\lim_{i \rightarrow \infty} j_i \rightarrow 0$, however.

Example 6.7. A quadratic perturbation

$$H = \sum_i \left(\frac{1}{2m} p_i^2 + \frac{m\omega_i^2}{2} x_i^2 \right) + g \sum_i \frac{m\omega_i^2}{2} x_i^2 \quad (6.67)$$

can also be diagonalized exactly by *Bogolyubov transformations*

$$\begin{pmatrix} a_i \\ a_i^* \end{pmatrix} \mapsto \begin{pmatrix} u & v \\ \bar{u} & \bar{v} \end{pmatrix} \begin{pmatrix} a_i \\ a_i^* \end{pmatrix} \quad (6.68)$$

with $|u|^2 - |v|^2 = 1$ and the intertwining unitary operators exist for each mode, but the product does not exist.

These examples show how important it is, to solve the linear and quadratic interactions *exactly* before attempting perturbation theory for the remaining nonlinear interactions. Of course, the perturbation series for the remaining terms can still diverge and the intertwining operators can fail to exist even if the series converges.

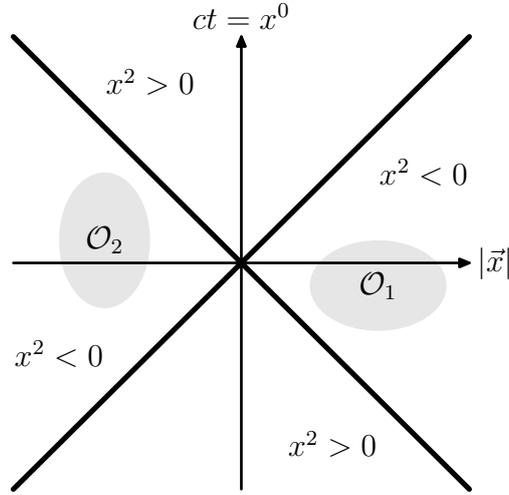


Figure 6.1: The light cone $x^2 = c^2t^2 - \vec{x}^2 = 0$ separates time-like $x^2 > 0$ from space-like $x^2 < 0$ distances in $\mathbf{R} \times \mathbf{R}^n$. The regions \mathcal{O}_1 and \mathcal{O}_2 are separated by a space-like distance.

6.3 Local Quantum Theory

While locality in concert with translation invariance provided the ingredients for Haag's theorem, it is nevertheless an important physical principle, realized in most models of microphysics.

A systematic approach considers the C^* -algebras $\mathcal{A}(\mathcal{V})$ and $\mathcal{A}(\mathcal{O})$ generated by the observables localized in a volume $\mathcal{V} \subseteq \mathbf{R}^n$ or a space-time region $\mathcal{O} \subseteq \mathbf{R} \times \mathbf{R}^n$. The algebras obviously must satisfy

$$\forall \mathcal{O}_1 \subseteq \mathcal{O}_2 : \mathcal{A}(\mathcal{O}_1) \subseteq \mathcal{A}(\mathcal{O}_2). \quad (6.69)$$

In relativistic physics, the measurements of observables in regions \mathcal{O}_1 and \mathcal{O}_2 that are separated by a space-like distance, as in figure 6.1, *must not* influence each other, i. e. the corresponding algebras must commute

$$[\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)] = 0. \quad (6.70)$$

This usually called *Einstein locality*.

In any case, the physically relevant algebra is the *local algebra*

$$\mathcal{A}_L = \bigcup_{\mathcal{O}} \mathcal{A}(\mathcal{O}) \quad (6.71)$$

or $\bigcup_{\mathcal{V}} \mathcal{A}(\mathcal{V})$, respectively. While the multiplication of two elements from different algebras $\mathcal{A}(\mathcal{O}_1)$ and $\mathcal{A}(\mathcal{O}_2)$ is *a priori* not defined, this is never a problem in practice when the elements of the algebras are generated by smearings of local operators, e. g.

$$\exp\left(i \int dx \psi(x) \bar{f}(x)\right) \quad (6.72)$$

with $\text{supp}(f) \in \mathcal{O}_i$. But note that \mathcal{A}_L is *not* complete by construction and one needs to complete it in an appropriate topology to the *local algebra*

$$\mathcal{A} = \overline{\mathcal{A}_L}, \quad (6.73)$$

Lecture 23: Wed, 16.01.2019

In relativistic physics, the finite propagation speed causes the completion in the norm topology to be invariant under the time evolution described by a one parameter family of automorphisms

$$\begin{aligned} \alpha : \mathbf{R} \times \mathcal{A} &\rightarrow \mathcal{A} \\ (t, A) &\mapsto \alpha_t(A), \end{aligned} \quad (6.74)$$

but in the non-relativistic approximation with instantaneous interactions, it can happen that only a larger algebra, obtained by completing in a weaker topology, is stable under the time evolution α .

6.3.1 Asymptotic Abelianness

In general, even in the non-relativistic case, we should be able to *require* that two observables commute asymptotically, if we move them apart

$$\forall A, B \in \mathcal{A} : \lim_{|x| \rightarrow \infty} [\alpha_x(A), B] = 0, \quad (6.75)$$

where α_x is the automorphism (6.54) of \mathcal{A} that realize spacial translations. Indeed, if *asymptotic abelianness* (6.75) was not satisfied, observables would be influenced by measurements at spacial infinity, which would be unphysical, of course.

However, one might relax (6.75) to *weak asymptotic abelianness*

$$\forall A, B \in \mathcal{A} : \text{w-lim}_{|x| \rightarrow \infty} [\pi(\alpha_x(A)), \pi(B)] = 0, \quad (6.76)$$

for any “relevant” representation (\mathcal{H}, π) of \mathcal{A} .

6.3.2 Physically Relevant Representations

The local algebra \mathcal{A} necessarily corresponds to a Weyl or Heisenberg algebra with an infinite number of **d.o.f.**, since the creation and annihilation operators are convolutions of field operators with functions

$$a(f) = \int dx \bar{f}(x)\psi(x). \quad (6.77)$$

Thus we cannot rely on the Stone-von Neumann theorem to make sure that predictions are independent of the representation chosen.

Therefore we must develop criteria to choose the correct representation(s) (\mathcal{H}, π) describing a given system in a given state:

1. *Existence of energy and momentum:* the space translation $\mathbf{R}^n \times \mathcal{A} \rightarrow \mathcal{A}$ and time evolution $\mathbf{R} \times \mathcal{A} \rightarrow \mathcal{A}$ automorphisms are realized by strongly continuous abelian groups of unitary operators

$$\forall x \in \mathbf{R}^n, A \in \mathcal{A} : U(x)\pi(A)U^*(x) = \pi(\alpha_x(A)) \quad (6.78a)$$

$$\forall y \in \mathbf{R}, A \in \mathcal{A} : U(t)\pi(A)U^*(t) = \pi(\alpha_t(A)). \quad (6.78b)$$

By Stone's theorem, this guarantees that there are self-adjoint generators P and H with a common dense domain $D \subset \mathcal{H}$ and corresponding to momentum and energy, respectively. Note that in condensed matter theory, the continuous group of translations might have to be replaced by the discrete subgroup of lattice translations.

2. *Stability:* the spectrum $\sigma(H)$ of the Hamiltonian is bounded from below

$$\sigma(H) \subseteq [E_{\min}, \infty) \quad (6.79)$$

and by the redefinition $U(t) \mapsto U(t)e^{-iE_{\min}t}$, we may choose $E_{\min} = 0$. The relativistically invariant form of the spectral condition is

$$\sigma(H) \subseteq [0, \infty) \quad (6.80a)$$

$$H^2 \geq P^2. \quad (6.80b)$$

3. *Existence of a ground state:* The infimum of $\sigma(H)$ is a non degenerate eigenvalue of H with associated eigenvector $\Omega \in \mathcal{H}$, which is called the *ground state* and has the properties:

- (a) Ω is cyclic **wrt** to the local algebra \mathcal{A}_L , i. e.

$$\mathcal{H} = \overline{\pi(\mathcal{A}_L)\Omega}. \quad (6.81)$$

and

(b) Ω is the unique translation invariant state.

Note that the non-degeneracy of the lowest eigenvalue is refers to the chosen representation (\mathcal{H}, π) *only*. There is in general more than one representation with the same local eigenvalue and corresponding ground state.

The cyclicity of Ω wrt \mathcal{A}_L is equivalent to the requirement that it must be possible, at least in principle, to prepare *all* states by local operations. The translation invariance of the ground state means that the large distance behaviour of the system is encoded in the ground state.

In fact, the uniqueness of the translation invariant state follows from asymptotic abelianness

Theorem 6.8. *In any irreducible cyclic representation $(\mathcal{H}, \pi, \Omega)$ of the algebra \mathcal{A} , that satisfies weak asymptotic abelianness (6.76), the ground state is the unique translation invariant state.*

This can be shown using

Theorem 6.9 (Von Neumann bicommutant theorem). *For any unital $*$ -subalgebra \mathcal{A} of $\mathcal{L}(\mathcal{H})$, the following are equivalent*

1. $\mathcal{A} = \mathcal{A}'' = (\mathcal{A}')'$
2. \mathcal{A} is closed in the weak topology
3. \mathcal{A} is closed in the strong topology,

which we state without proof.

Proof of theorem 6.8. If there is another translation invariant state not equal to Ω , we can construct a state $\Omega' \in \mathcal{H}$ that is orthogonal to Ω . Then, with P_Ω the projection on Ω ,

$$\begin{aligned} \forall A \in \mathcal{A} : (\Omega', \pi(A)\Omega) &= (\Omega', \pi(\alpha_x(A))\Omega) = (\Omega', \pi(\alpha_x(A))P_\Omega\Omega) \\ &= \underbrace{(\Omega', P_\Omega\pi(\alpha_x(A))\Omega)}_{= 0} + \underbrace{(\Omega', [\pi(\alpha_x(A)), P_\Omega]\Omega)}_{\text{independent of } x} \end{aligned} \quad (6.82)$$

because $P_\Omega\Omega' = 0$ and both $(\Omega', \pi(\alpha_x(A))\Omega)$ and $(\Omega, \pi(\alpha_x(A))\Omega)$ are independent of x . Therefore

$$\forall A \in \mathcal{A} : (\Omega', \pi(A)\Omega) = \lim_{|x| \rightarrow \infty} (\Omega', [\pi(\alpha_x(A)), P_\Omega]\Omega) . \quad (6.83)$$

From the bicommutant theorem 6.9

$$\overline{\pi(\mathcal{A})} = \left(\overline{\pi(\mathcal{A})} \right)'' . \quad (6.84)$$

OTOH, in an irreducible representation (\mathcal{H}, π) , $\pi(\mathcal{A})' = \{\lambda \mathbf{1}\}_{\lambda \in \mathbb{C}}$ and therefore

$$\pi(\mathcal{A})'' = \mathcal{L}(\mathcal{H}) . \quad (6.85)$$

Finally

$$\left(\overline{\pi(\mathcal{A})} \right)'' \supseteq \pi(\mathcal{A})'' \quad (6.86)$$

since $\overline{\pi(\mathcal{A})} \supseteq \pi(\mathcal{A})$. Thus we have shown

$$\overline{\pi(\mathcal{A})} \supseteq \mathcal{L}(\mathcal{H}) \quad (6.87)$$

and therefore

$$P_\Omega \in \overline{\pi(\mathcal{A})} . \quad (6.88)$$

From asymptotic abelianness we can now argue

$$\forall A \in \mathcal{A} : \lim_{|x| \rightarrow \infty} (\Omega', [\pi(\alpha_x(A)), P_\Omega] \Omega) = 0 \quad (6.89)$$

and finally

$$(\Omega', \pi(\mathcal{A})\Omega) = 0 , \quad (6.90)$$

i. e. $\Omega' = 0$ from irreducibility of $(\mathcal{H}, \pi, \Omega)$. \square

Actually, using an almost identical calculation, one can prove an even stronger result:

$$\text{w-lim}_{|x| \rightarrow \infty} \pi(\alpha_x(A)) = (\Omega, \pi(A)\Omega) \mathbf{1} . \quad (6.91)$$

Indeed

$$\begin{aligned} \forall B \in \mathcal{A} : \text{w-lim}_{|x| \rightarrow \infty} \pi(\alpha_x(A))\pi(B)\Omega &= \text{w-lim}_{|x| \rightarrow \infty} \pi(B)\pi(\alpha_x(A))\Omega \\ &= \pi(B) \text{w-lim}_{|x| \rightarrow \infty} ([\pi(\alpha_x(A)), P_\Omega] + P_\Omega\pi(\alpha_x(A))) \Omega \\ &= \pi(B)P_\Omega\pi(A)\Omega = \pi(B)\Omega(\Omega, \pi(A)\Omega) . \end{aligned} \quad (6.92)$$

We arrive at the conclusion, that under the assumption of asymptotic abelianness, the large distance behaviour of the system is encoded entirely in the ground state.

6.3.3 Cluster Property

Using

$$\lim_{|x| \rightarrow \infty} (\Omega, \pi(\alpha_x(A))\pi(B)\Omega) = (\Omega, (\Omega, \pi(A)\Omega)\pi(B)\Omega) = (\Omega, \pi(A)\Omega)(\Omega, \pi(B)\Omega) \quad (6.93)$$

we have just shown

Theorem 6.10 (Cluster property). *In any irreducible cyclic representation $(\mathcal{H}, \pi, \Omega)$ of the algebra \mathcal{A} , that satisfies weak asymptotic abelianness (6.76), the ground state correlations factorize for infinite spacial separation*

$$\forall A, B \in \mathcal{A} : \lim_{|x| \rightarrow \infty} (\Omega, \pi(\alpha_x(A))\pi(B)\Omega) = (\Omega, \pi(A)\Omega)(\Omega, \pi(B)\Omega). \quad (6.94)$$

This cluster property is crucially important for the physical interpretation of quantum theory:

- it allows the definition of scattering states of separated particles and thus of an S -matrix,
- it shows that the necessarily non-local character of the translation invariant ground state only appears in expectation values of observables and that all correlations among spacially separated observables vanish, if their distance is taken to infinity

In fact, one could also start from the cluster property, which is technically simpler to handle and derive irreducibility and uniqueness of the translation invariant state from it.

6.4 Free Bose Gas

A “gas” of interacting nonrelativistic bosons in a volume $V \subset \mathbf{R}^n$ is described by the Hamiltonian

$$H_V = \underbrace{\frac{1}{2m} \int_V dx (\nabla\psi^*(x)) (\nabla\psi(x))}_{\sim \frac{1}{2m} p^2} + \frac{1}{2} \int_V dx dy \psi^*(x)\psi^*(y) H_{\text{int.}}(x-y)\psi(x)\psi(y)$$

with ψ and ψ^* annihilation and creation operators². respectively. Already the non-interacting case $H_{\text{int.}} = 0$ with finite occupation number and energy density is quite interesting and we shall concentrate on it

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$$H_V = \frac{1}{2m} \int_V dx (\nabla\psi^*(x)) (\nabla\psi(x)) \quad (6.95)$$

and its infinite volume limit

$$H = \lim_{V \rightarrow \mathbf{R}^n} H_V = \frac{1}{2m} \int dx (\nabla\psi^*(x)) (\nabla\psi(x)) . \quad (6.96)$$

The occupation number, **a.k.a.** particle number, operator is

$$N_V = \int_V dx \psi^*(x)\psi(x) \quad (6.97)$$

and its density in the infinite volume limit

$$n = \lim_{V \rightarrow \mathbf{R}^n} \frac{N_V}{V} = \lim_{V \rightarrow \mathbf{R}^n} \frac{1}{V} \int_V dx \psi^*(x)\psi(x) . \quad (6.98)$$

6.4.1 Heuristic Discussion

Below, we will discuss the model in greater mathematical rigor, but we start with a heuristic discussion of what to expect. Both (6.95) and (6.96) are formally³ positive and a state Ω with

$$H_V\Omega = 0 , \quad (6.99)$$

if it exists and is unique, is necessarily the ground state of the system. Thus

$$0 = (\Omega, H_V\Omega) = \frac{1}{2m} \int_V dx (\nabla\psi(x)\Omega, \nabla\psi(x)\Omega) \quad (6.100)$$

and

$$\nabla\psi(x)\Omega = 0 . \quad (6.101)$$

Since the momentum operator P generates translations

$$\nabla\psi(x) = i [P, \psi(x)] , \quad (6.102)$$

²Operator valued distributions, to be precise

³A product of operator valued distributions like $(\nabla\psi^*(x)) (\nabla\psi(x))$ is not well defined and we must show that the integral exists and leads to a positive operator.

we infer from the translation invariance of the ground state $P\Omega = 0$ that

$$0 = [P, \psi(x)]\Omega = P\psi(x)\Omega, \quad (6.103)$$

i. e. that $\psi(x)\Omega$ is translation invariant itself. However there is only one translation invariant state and therefore

$$\exists c \in \mathbf{C} : \psi(x)\Omega = c\Omega. \quad (6.104)$$

Obviously, we can compute c as follows

$$c = (\Omega, \psi(x)\Omega) =: \langle \psi \rangle. \quad (6.105)$$

We can define operators

$$\psi_F(x) = \psi(x) - \langle \psi \rangle \mathbf{1} \quad (6.106a)$$

$$\psi_F^*(x) = \psi^*(x) - \overline{\langle \psi \rangle} \mathbf{1} \quad (6.106b)$$

that satisfy the *same* commutation relations as $\psi(x)$ and $\psi^*(x)$, i. e. furnish another representation of the same algebra. From

$$\psi_F(x)\Omega = 0 \quad (6.107)$$

we see that it is a Fock representation. The correlation functions turn out to be very simple and free of singularities

$$\begin{aligned} \langle \psi^*(x)\psi(y) \rangle &:= (\Omega, \psi^*(x)\psi(y)\Omega) = (\psi(x)\Omega, \psi(y)\Omega) \\ &= (\langle \psi \rangle \Omega, \langle \psi \rangle \Omega) = \overline{\langle \psi \rangle} \langle \psi \rangle (\Omega, \Omega) = |\langle \psi \rangle|^2 \end{aligned} \quad (6.108a)$$

and

$$\langle \psi(x)\psi(y) \rangle = \langle \psi \rangle^2 \quad (6.108b)$$

$$\langle \psi^*(x)\psi^*(y) \rangle = \left(\overline{\langle \psi \rangle} \right)^2. \quad (6.108c)$$

From the definition we have a well defined occupation number density

$$n = \frac{N_V}{V} = \langle \psi^*(x)\psi(x) \rangle = |\langle \psi \rangle|^2 \quad (6.109)$$

and we can parametrize the ground state $\Omega_{n,\theta}$ by two parameters:

$$\exists \theta \in [0, 2\pi) : \langle \psi \rangle = (\Omega_{n,\theta}, \psi(x)\Omega_{n,\theta}) = \sqrt{n}e^{i\theta}, \quad (6.110)$$

However, while the number density n is fixed by the preparation of the system, the phase θ is free. This situation reminds us of the particle on a circle, as discussed in section 5.3.3. Again we have a family of representations, labelled by a phase $\theta \in [0, 2\pi)$. Indeed one can define states

$$\begin{aligned} \omega_{n,\theta} : \mathcal{A} &\rightarrow \mathbf{C} \\ A &\mapsto (\Omega_{n,\theta}, \pi_{n,\theta}(A)\Omega_{n,\theta}) \end{aligned} \quad (6.111)$$

and in the infinite volume limit $|V| \rightarrow \infty$, the GNS-representations constructed from $\omega_{n,\theta}$ and $\omega_{n,\theta'}$ will be unitarily inequivalent for $n > 0 \wedge \theta \neq \theta'$.

The classical theory (6.96) has a rigid *gauge symmetry*⁴

$$\begin{pmatrix} \psi(x) \\ \bar{\psi}(x) \end{pmatrix} \mapsto \begin{pmatrix} e^{i\theta}\psi(x) \\ e^{-i\theta}\bar{\psi}(x) \end{pmatrix} \quad (6.112)$$

that leaves the Hamiltonian invariant. Also the algebra generated by the observables has the corresponding one parameter group of *-automorphisms

$$\begin{aligned} \beta : [0, 2\pi) \times \mathcal{A} &\rightarrow \mathcal{A} \\ (\theta, \psi(x)) &\mapsto e^{i\theta}\psi(x), \end{aligned} \quad (6.113)$$

but, as we will show below explicitly, the symmetry can not be realized by unitary operators in representations with $n > 0$. Thus the gauge symmetry is a *spontaneously broken symmetry*.

6.4.2 Tensor Products of Hilbert Spaces

Before embarking on the detailed discussion, we should clarify some important subtleties in the construction of the tensor products of Hilbert spaces that will be used below⁵.

⁴NB: this is *not* the *gauge invariance*

$$\begin{pmatrix} \psi(x) \\ \bar{\psi}(x) \\ A_\mu(x) \end{pmatrix} \mapsto \begin{pmatrix} e^{i\theta(x)}\psi(x) \\ e^{-i\theta(x)}\bar{\psi}(x) \\ A_\mu(x) + \partial_\mu\theta(x) \end{pmatrix}$$

of Quantum Electrodynamics with a space-time dependent function θ , but a constant phase transformation. Note that *gauge invariance* is not a symmetry of a model, but a reparametrisation of our mathematical description of the same physical state. **OTOH**, rigid phase transformations relate different physical states and are symmetries, **iff** the dynamics respect them.

⁵NB: this section was added only *after* questions came up in lecture #24.

Finite Tensor Products

A concise definition of the tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ of two Hilbert spaces $\mathcal{H}_{1,2}$ that doesn't refer to the choice of a basis goes back to Murray and von Neumann [MvN36]. Given two vectors $\phi_i \in \mathcal{H}_i$ with $i = 1, 2$, we *define* the product $\phi_1 \otimes \phi_2$ as an anti-linear functional on the cartesian product of the Hilbert spaces

$$\begin{aligned} \phi_1 \otimes \phi_2 : \mathcal{H}_1 \times \mathcal{H}_2 &\rightarrow \mathbf{C} \\ \langle \psi_1, \psi_2 \rangle &\mapsto \overline{(\phi_1, \psi_1)}(\phi_2, \psi_2) = (\psi_1, \phi_1)(\psi_2, \phi_2) \end{aligned} \quad (6.114)$$

where we have used $\langle \cdot, \cdot \rangle$ for the pairs that make up the cartesian product to avoid confusion with the inner product (\cdot, \cdot) . Obviously, linear combinations of these functionals form a vector space \mathcal{E} and we can define an sesquilinear inner product extending

$$\begin{aligned} (\cdot, \cdot) : \mathcal{E} \times \mathcal{E} &\rightarrow \mathbf{C} \\ \langle \phi_1 \otimes \phi_2, \phi'_1 \otimes \phi'_2 \rangle &\mapsto (\phi_1, \phi'_1)(\phi_2, \phi'_2). \end{aligned} \quad (6.115)$$

by linearity. It is easy to show (cf. [RS80], sect. II.4) that this inner product is well defined (i. e. vanishes if one argument is a sum of vectors adding up to the zero vector) and positive definite. The tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ is then the completion of the vector space \mathcal{E} wrt this inner product.

This definition extends easily to the tensor product of any finite family of Hilbert spaces.

Infinite Tensor Products

Infinite Tensor Products (ITPs) are more subtle. They were introduced and studied in detail by John von Neumann [vN39]. He introduced the distinction between the *complete ITP* that leads to a non-separable Hilbert space and *incomplete ITPs* which are separable Hilbert spaces.

The construction starts as in the finite dimensional case from anti-linear functionals

$$\Phi \in \widehat{\bigotimes_{n \in I} \mathcal{H}_n} \quad (6.116)$$

that are defined on the sequences $\{\psi_n\}_{n \in I}$ of vectors $\psi_n \in \mathcal{H}_n$ with

$$\prod_{n \in I} \|\psi_n\| < \infty. \quad (6.117)$$

A subspace

$$\Phi \in \widetilde{\bigotimes_{n \in I} \mathcal{H}_n} \subseteq \widehat{\bigotimes_{n \in I} \mathcal{H}_n} \quad (6.118)$$

are those anti-linear functionals defined as

$$\Phi : \{\psi_n\}_{n \in I} \mapsto \prod_{n \in I} (\psi_n, \phi_n) \quad (6.119)$$

with

$$\prod_{n \in I} \|\phi_n\| < \infty \quad (6.120)$$

and their finite linear combinations. The complete **ITP** is the completion of $\widetilde{\bigotimes}_{n \in I} \mathcal{H}_n$ in a natural Hilbert space topology

$$\bigotimes_{n \in I} \mathcal{H}_n = \overline{\left(\widetilde{\bigotimes}_{n \in I} \mathcal{H}_n \right)} \subseteq \widehat{\bigotimes}_{n \in I} \mathcal{H}_n \quad (6.121)$$

(see [vN39] for the details).

One can define equivalence classes Ψ of sequences $\{\psi_n\}_{n \in I}$ with

$$\sum_{n \in I} \left| \|\psi_n\| - 1 \right| < \infty \quad (6.122)$$

as

$$\{\psi_n\}_{n \in I} \sim \{\phi_n\}_{n \in I} \iff \sum_{n \in I} |(\psi_n, \phi_n) - 1| < \infty. \quad (6.123)$$

Given such an equivalence class Ψ , we can define the incomplete **ITP**

$$\bigotimes_{n \in I}^{\Psi} \mathcal{H}_n = \left\{ \bigotimes_{n \in I} \psi_n : \{\psi_n\}_{n \in I} \in \Psi \right\} \subseteq \bigotimes_{n \in I} \mathcal{H}_n. \quad (6.124)$$

One can show [vN39] that vectors from the incomplete **ITP**s of different equivalence classes are mutually orthogonal and that their combination yields the complete **ITP**.

Below, we will use the incomplete **ITP** for the equivalence class that contains the cyclic vector Ω and denote it by the vector instead of its equivalence class

$$\bigotimes_{n \in I}^{\Omega} \mathcal{H}_n. \quad (6.125)$$

6.4.3 Detailed Discussion

Canonical Commutation Relations

The C^* -algebra \mathcal{A}_V generated by the observables of the free bose gas in a volume V (including $V \rightarrow \mathbf{R}^n$) is a Weyl system. Instead of the unbounded

operator valued distributions ψ and ψ^* , we introduce unitary algebra elements $U(f)$, $V(g)$ with test functions $f, g : V \rightarrow \mathbf{R}$ and multiplication laws

$$U(f)V(g) = V(g)U(f)e^{-i\langle f, g \rangle} \quad (6.126a)$$

$$U(f)U(g) = U(f + g) \quad (6.126b)$$

$$V(f)V(g) = V(f + g), \quad (6.126c)$$

where

$$\langle f, g \rangle = \int dx f(x)g(x). \quad (6.127)$$

As discussed above, these relations are compatible with

$$U(f) = e^{i\phi(f)} \quad (6.128a)$$

$$V(g) = e^{i\pi(g)} \quad (6.128b)$$

and an unbounded self-adjoint algebra elements

$$[\phi(f), \pi(g)] = i\langle f, g \rangle \quad (6.129a)$$

$$[\phi(f), \phi(g)] = [\pi(f), \pi(g)] = 0. \quad (6.129b)$$

We can now introduce

$$a(f) = \frac{1}{\sqrt{2}} (\phi(f) + i\pi(f)) \quad (6.130a)$$

$$a^*(f) = \frac{1}{\sqrt{2}} (\phi(f) - i\pi(f)) \quad (6.130b)$$

and find the commutation relations

$$[a(f), a^*(g)] = \langle f, g \rangle \quad (6.131a)$$

$$[a(f), a(g)] = [a^*(f), a^*(g)] = 0. \quad (6.131b)$$

These are in turn compatible with

$$a(f) = \int_V dx f(x)\psi(x) \quad (6.132a)$$

$$a^*(f) = \int_V dx f(x)\psi^*(x) \quad (6.132b)$$

and the formal commutation relations

$$[\psi(x), \psi^*(y)] = \delta(x - y) \quad (6.133a)$$

$$[\psi(x), \psi(y)] = [\psi^*(x), \psi^*(y)] = 0, \quad (6.133b)$$

i. e. our local annihilation and creation operators.

Representations

Our algebra \mathcal{A} is the completion of linear combinations of products of $U(f)$ s and $V(g)$ s. Using the canonical commutation relations (6.126), we can bring every product in the form $e^{i\phi}U(f)V(g)$ with $\phi \in [0, 2\pi)$. Thus, it suffices to specify the value of a state $\omega : \mathcal{A} \rightarrow \mathbf{C}$ on such elements to define a state on the whole algebra by linearity. Since a state ω determines a corresponding GNS-representation up to unitary equivalence, we can characterize a representation with cyclic vector Ω by the function

$$\omega_\Omega(f, g) = (\Omega, \pi(U(f))\pi(V(g))\Omega) . \quad (6.134)$$

Fock Representation

In order to avoid the unbounded annihilation and creation operators, we don't define the Hilbert space \mathcal{H}_F of the Fock representation $(\mathcal{H}_F, \pi_F, \Omega_F)$ as the direct sum of n -particle spaces. Instead [AW63], we use the *incomplete ITP*

$$\mathcal{H}_F = \bigotimes_{i \in \mathbf{N}}^{\Omega_F} \mathcal{H}_i \quad (6.135)$$

of Hilbert spaces $\{\mathcal{H}_i\}_{i \in \mathbf{N}}$ containing Ω_F in which self-adjoint operators Q_i and P_i with **CCR** $[Q_i, P_i] = i$ act irreducibly and Ω_i are cyclic vectors⁶. For convenience, we will however drop the Ω_F from the tensor product symbol. The cyclic vector Ω_F is accordingly

$$\Omega_F = \bigotimes_{i \in \mathbf{N}} \Omega_i . \quad (6.136)$$

Define unitary operators on \mathcal{H}_F by

$$U_i(\xi) = \mathbf{1}_1 \otimes \dots \otimes \mathbf{1}_{i-1} \otimes e^{i\xi Q_i} \otimes \mathbf{1}_{i+1} \otimes \dots \quad (6.137a)$$

$$V_i(\eta) = \mathbf{1}_1 \otimes \dots \otimes \mathbf{1}_{i-1} \otimes e^{i\eta P_i} \otimes \mathbf{1}_{i+1} \otimes \dots . \quad (6.137b)$$

Choosing a complete orthonormal basis $\{f_i\}_{i \in \mathbf{N}}$ for $L^2(V, \mathbf{R})$, we can decompose every function $f \in L^2(V, \mathbf{R})$

$$f = \sum_{i \in \mathbf{N}} \lambda_i^f f_i \quad (6.138)$$

with real coefficients $\{\lambda_i\}_{i \in \mathbf{N}}$ and define unitary operators

$$U_F(f) = \prod_{i \in \mathbf{N}} U_i(\lambda_i^f) = \bigotimes_{i \in \mathbf{N}} e^{i\lambda_i^f Q_i} \quad (6.139a)$$

⁶In the Schrödinger representation, $x_i \rightarrow e^{-x_i^2/2}/\pi^{1/4}$ is a good choice for Ω_i , but any square integrable function that is almost everywhere non zero is fine.

$$V_F(g) = \prod_{i \in \mathbf{N}} V_i(\lambda_i^g) = \bigotimes_{i \in \mathbf{N}} e^{i\lambda_i^g P_i} \quad (6.139b)$$

for every $f \in L^2(V, \mathbf{R})$. By construction, the $U_F(f)$, $V_F(f)$ are irreducible and Ω_F is cyclic.

We can now compute

$$\begin{aligned} \omega_F(f, g) &= \omega_{\Omega_F}(f, g) = (\Omega_F, U_F(f)V_F(g)\Omega_F) \\ &= \exp\left(-\frac{1}{4}\langle f, f \rangle - \frac{1}{4}\langle g, g \rangle - \frac{i}{2}\langle f, g \rangle\right) \end{aligned} \quad (6.140)$$

using (6.138) and

$$\langle f, g \rangle = \int_V dx f(x)g(x) = \sum_{i \in \mathbf{N}} \lambda_i^f \lambda_i^g. \quad (6.141)$$

Using the strongly continuous groups of unitary operators $\{U_F(sf)\}_{s \in \mathbf{R}}$ and $\{V_F(sg)\}_{s \in \mathbf{R}}$, we can construct the generators

$$\phi_F(f) = -i \left. \frac{d}{ds} \right|_{s=0} U_F(sf) \quad (6.142a)$$

$$\pi_F(g) = -i \left. \frac{d}{ds} \right|_{s=0} V_F(sg). \quad (6.142b)$$

Note that the functional form of ω_F (6.140) suffices to calculate expectation values of polynomials in these generators, e. g.

$$\begin{aligned} (\Omega_F, \phi_F(f_1)\phi_F(f_2)\Omega_F) &= - \left. \frac{d}{ds_1} \frac{d}{ds_2} \right|_{s_1=s_2=0} (\Omega_F, U_F(s_1 f_1)U_F(s_2 f_2)\Omega_F) \\ &= - \left. \frac{d}{ds_1} \frac{d}{ds_2} \right|_{s_1=s_2=0} (\Omega_F, U_F(s_1 f_1 + s_2 f_2)\Omega_F) \\ &= - \left. \frac{d}{ds_1} \frac{d}{ds_2} \right|_{s_1=s_2=0} \omega_F(s_1 f_1 + s_2 f_2, 0) \\ &= - \left. \frac{d}{ds_1} \frac{d}{ds_2} \right|_{s_1=s_2=0} \exp\left(-\frac{1}{4}\langle s_1 f_1 + s_2 f_2, s_1 f_1 + s_2 f_2 \rangle\right) \\ &= \frac{1}{2}\langle f_1, f_2 \rangle. \end{aligned} \quad (6.143)$$

This is the same as the result obtained from formal manipulations of the unbounded annihilation and creation operators

$$\begin{aligned}
(\Omega_F, \phi_F(f_1)\phi_F(f_2)\Omega_F) &= \frac{1}{2} (\Omega_F, (a(f_1) + a^*(f_1)) (a(f_2) + a^*(f_2)) \Omega_F) \\
&= \frac{1}{2} (\Omega_F, a(f_1)a^*(f_2)\Omega_F) = \frac{1}{2} (\Omega_F, [a(f_1), a^*(f_2)]\Omega_F) = \frac{1}{2} \langle f_1, f_2 \rangle. \quad (6.144)
\end{aligned}$$

Note also that the representation carries a unitary representation of the *Euclidean group*

$$\alpha_{R,a}(U_F(f)) = U_F(\alpha_{R,a}^* f) \quad (6.145a)$$

$$\alpha_{R,a}(V_F(g)) = V_F(\alpha_{R,a}^* g) \quad (6.145b)$$

with

$$(\alpha_{R,a}^* f)(x) = f(R^{-1}(x - a)). \quad (6.146)$$

Occupation Number Density

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As mentioned above, a rigorous definition of the occupation number operator

$$N_V = \int_V dx \psi^*(x)\psi(x) \quad (6.97)$$

is subtle, because $\psi(x)$ and $\psi^*(x)$ are *not* operators that can be multiplied. Instead ψ and ψ^* must be understood as operator valued distributions. A rigorous definition uses a complete orthonormal set $\{f_i\}_{i \in \mathbf{N}}$ for $V \rightarrow \mathbf{R}$ (including $V = \mathbf{R}^n$)

$$N_V = \sum_{i \in \mathbf{N}} a^*(f_i)a(f_i). \quad (6.147)$$

However, N_V and N_V/V are unbounded operators, so the definition of the infinite volume limits for occupation number and occupation number density

$$N = \lim_{V \rightarrow \mathbf{R}^n} N_V \quad (6.148a)$$

$$n = \lim_{V \rightarrow \mathbf{R}^n} \frac{N_V}{V} \quad (6.148b)$$

is not obvious. We can however use the convergence of a group of unitary exponentials to define the convergence of the generators:

Definition 6.11. If $\{T_n\}_{n \in \mathbf{N}}$ and T are self-adjoint operators on a Hilbert space \mathcal{H} , we say that

$$\lim_{n \rightarrow \infty} T_n = T \quad (6.149)$$

iff

$$\forall \lambda \in \mathbf{R} : \lim_{n \rightarrow \infty} e^{i\lambda T_n} = e^{i\lambda T}. \quad (6.150)$$

The operators $\phi(f)$ and $\pi(g)$ are linear combinations of $a(f)$ and $a^*(f)$ with

$$[N_V, a(f)] = -a(f) \quad (6.151a)$$

$$[N_V, a^*(f)] = a^*(f). \quad (6.151b)$$

Consequently all polynomials in $\phi(f)$ and $\pi(g)$ can change the eigenvalues of N_V only by a *finite* number. In states with a non-vanishing occupation number density n (cf. (6.148b)), the occupation number (6.148a) is formally infinite. Therefore, polynomials in $\phi(f)$ and $\pi(g)$ will *not* be able to change n

$$[n, \phi(f)] = [n, \pi(g)] = 0. \quad (6.152)$$

This observation can be turned into a

Lemma 6.12. *In a irreducible representation (\mathcal{H}, π) , the occupation number density n is a constant, if it exists.*

Proof. Using our notion of convergence, we must show that

$$\forall f, g : \forall \lambda \in \mathbf{R} : \lim_{V \rightarrow \mathbf{R}^n} [U(f), e^{i\lambda N_V/V}] = \lim_{V \rightarrow \mathbf{R}^n} [V(f), e^{i\lambda N_V/V}] = 0. \quad (6.153)$$

Indeed, by a straightforward calculation (\rightarrow *exercise*)

$$e^{i\lambda N_V/V} U(f) = \underbrace{U\left(f \cos \frac{\lambda}{V}\right)}_{\rightarrow U(f)} \underbrace{V\left(f \sin \frac{\lambda}{V}\right) e^{i\frac{\langle f, f \rangle}{4} \sin \frac{2\lambda}{V}}}_{\rightarrow \mathbf{1}} e^{i\lambda N_V/V} \quad (6.154a)$$

$$e^{i\lambda N_V/V} V(g) = \underbrace{V\left(g \cos \frac{\lambda}{V}\right)}_{\rightarrow V(g)} \underbrace{U\left(-g \sin \frac{\lambda}{V}\right) e^{i\frac{\langle g, g \rangle}{4} \sin \frac{2\lambda}{V}}}_{\rightarrow \mathbf{1}} e^{i\lambda N_V/V} \quad (6.154b)$$

and the product of two limits of unitary operators is the limit of the products, as will be shown in the following lemma. \square

Lemma 6.13. *Let*

$$\text{s-lim}_{n \rightarrow \infty} U_n = U \quad (6.155a)$$

$$\text{s-lim}_{n \rightarrow \infty} V_n = V \quad (6.155b)$$

with all U_n and V_n unitary, then

$$\text{s-lim}_{n \rightarrow \infty} U_n V_n = UV. \quad (6.156)$$

Proof. As the strong limits of unitary operators, U and V are isometries. Then

$$\forall \Psi \in \mathcal{H} : \|(U_n^{-1}U - \mathbf{1})\Psi\|^2 = 2(\|\Psi\|^2 - \operatorname{Re}(U\Psi, U_n\Psi)) \xrightarrow{n \rightarrow \infty} 0 \quad (6.157)$$

and therefore

$$\operatorname{s-lim}_{n \rightarrow \infty} U_n^{-1}U = \mathbf{1}. \quad (6.158)$$

Also

$$\forall \Psi \in \mathcal{H} : \|(U_n V_n - UV)\Psi\|^2 = 2(\|\Psi\|^2 - \operatorname{Re}(U_n^{-1}UV\Psi, V_n\Psi)) \xrightarrow{n \rightarrow \infty} 0 \quad (6.159)$$

using this result twice. \square

Lecture 26: Fri, 25.01.2019

Lemma 6.14. *In a cyclic representation (\mathcal{H}, π, Ψ) with*

$$\exists \rho \in \mathbf{R} : \forall \lambda \in \mathbf{R} : \lim_{V \rightarrow \mathbf{R}^n} e^{i\lambda N_V/V} \Psi = e^{i\lambda \rho} \Psi, \quad (6.160)$$

the occupation number density $n = \lim N_V/V$ exists and

$$n = \rho \mathbf{1}. \quad (6.161)$$

Proof.

$$\begin{aligned} \lim_{V \rightarrow \mathbf{R}^n} e^{i\lambda N_V/V} U(f)V(g)\Psi &= U(f)V(g) \lim_{V \rightarrow \mathbf{R}^n} e^{i\lambda N_V/V} \Psi \\ &= U(f)V(g)e^{i\lambda \rho} \Psi = e^{i\lambda \rho} U(f)V(g)\Psi \end{aligned} \quad (6.162)$$

and since linear combinations of $U(f)V(g)\Psi$ are dense in \mathcal{H} ,

$$\operatorname{s-lim}_{V \rightarrow \mathbf{R}^n} e^{i\lambda N_V/V} = e^{i\lambda \rho} \mathbf{1}. \quad (6.163)$$

\square

One can use similar arguments to show that the energy density of a free bose gas is also a constant.

Representation for Finite Occupation Number Density

We can construct an appropriate representation for finite occupation number density ρ by computing the limit

$$\omega^\rho(f, g) = \lim_{V \rightarrow \mathbf{R}} \omega_V^\rho(f, g) \quad (6.164)$$

of state with occupation number $\nu = \rho V$ in the finite volume Fock representation

$$\omega_V^\rho(f, g) = (\Psi_F^\rho, U_F(f) V_F(g) \Psi_F^\rho) . \quad (6.165)$$

The lowest energy wave function in the finite volume V is constant

$$\begin{aligned} f_V : V &\rightarrow \mathbf{R} \\ x &\mapsto \frac{1}{\sqrt{|V|}} \end{aligned} \quad (6.166)$$

and has no infinite volume limit. The Fock state with $\rho = \nu/|V|$ is

$$\Psi_F^\rho = \frac{1}{\sqrt{\nu!}} (a^*(f_V))^\nu \Omega_F \quad (6.167)$$

and therefore

$$\omega_V^{\nu/|V|}(f, g) = \frac{1}{\nu!} (\Omega_F, (a(f_V))^\nu U_F(f) V_F(g) (a^*(f_V))^\nu \Omega_F) . \quad (6.168)$$

One can compute (\rightarrow *exercise*)

$$\omega_V^{\nu/|V|}(f, g) = \omega_F(f, g) L_\nu \left(\frac{\langle f, f_V \rangle^2 + \langle g, f_V \rangle^2}{2} \right) \quad (6.169)$$

with L_ν the ν th Laguerre polynomial. Note that

$$\langle f, f_V \rangle = \int_V dx \frac{f(x)}{\sqrt{|V|}} = \sqrt{\frac{\rho}{\nu}} \int_V dx f(x) =: \sqrt{\frac{\rho}{\nu}} f_0 . \quad (6.170)$$

In the infinite volume limit $\nu \rightarrow \infty$ and $V \rightarrow \mathbf{R}^n$ with $\rho = \nu/|V|$ constant. In this limit, the Laguerre polynomial becomes a Bessel function

$$\lim_{\nu \rightarrow \infty} L_\nu \left(\frac{z}{\nu} \right) = J_0(2\sqrt{z}) . \quad (6.171)$$

(theorem 8.1.3 of [Sze75]) and we find

$$\omega^\rho(f, g) = \omega_F(f, g) J_0 \left(\sqrt{2\rho(f_0^2 + g_0^2)} \right) . \quad (6.172)$$

Since the Bessel function is not manifestly positive, it is not obvious that $\omega^\rho : \mathcal{A} \rightarrow \mathbf{C}$ defines a positive functional and that we can use the GNS construction to obtain a representation. A convenient way to prove the positivity is to construct a representation explicitly.

The factorized form of (6.172) suggests to use a tensor product

$$\mathcal{H}_\rho = \mathcal{H}_F \otimes \mathcal{K}, \quad (6.173)$$

with \mathcal{K} another Hilbert space. In fact, we can choose the square integrable real functions on the unit circle

$$\mathcal{K} = L^2(S^1, \mathbf{C}, dx/2\pi). \quad (6.174)$$

We define bounded operators S and C on \mathcal{K} by

$$(S\psi)(\theta) = \sin \theta \cdot \psi(\theta) \quad (6.175a)$$

$$(C\psi)(\theta) = \cos \theta \cdot \psi(\theta) \quad (6.175b)$$

and denote by $\Phi_0 \in \mathcal{K}$ the constant function

$$\begin{aligned} \psi : S^1 &\rightarrow \mathbf{C} \\ \theta &\mapsto 1. \end{aligned} \quad (6.176)$$

The ground state is

$$\Omega_\rho = \Omega_F \otimes \Phi_0 \quad (6.177)$$

and the Weyl operators are represented by

$$\begin{aligned} U_\rho(f) &= U_F(f) \otimes e^{i\sqrt{2\rho}f_0C} \\ V_\rho(g) &= V_F(g) \otimes e^{i\sqrt{2\rho}g_0S}, \end{aligned} \quad (6.178)$$

because the operators in the second factor form a two-parameter abelian group

$$e^{i\sqrt{2\rho}f_0C} e^{i\sqrt{2\rho}f'_0C} = e^{i\sqrt{2\rho}(f_0+f'_0)C} \quad (6.179a)$$

$$e^{i\sqrt{2\rho}g_0S} e^{i\sqrt{2\rho}g'_0S} = e^{i\sqrt{2\rho}(g_0+g'_0)S} \quad (6.179b)$$

$$e^{i\sqrt{2\rho}f_0C} e^{i\sqrt{2\rho}g_0S} = e^{i\sqrt{2\rho}(f_0C+g_0S)}. \quad (6.179c)$$

We can compute the expectation values and find

$$\begin{aligned} (\Omega_\rho, U_\rho(f)V_\rho(g)\Omega_\rho) &= \underbrace{(\Omega_F, U_F(f)V_F(g)\Omega_F)}_{\omega_F(f, g)} \underbrace{\int_0^{2\pi} \frac{d\theta}{2\pi} e^{i\sqrt{2\rho}(f_0 \cos \theta + g_0 \sin \theta)}}_{J_0\left(\sqrt{2\rho}(f_0^2 + g_0^2)\right)} \\ &= \underbrace{\omega_F(f, g) J_0\left(\sqrt{2\rho}(f_0^2 + g_0^2)\right)}_{\omega^\rho(f, g)} \end{aligned} \quad (6.180)$$

as desired.

It remains to show that Ω_ρ is indeed a cyclic vector. For this purpose, consider a family of functions

$$\{f_{s,\alpha} : \mathbf{R}^n \rightarrow \mathbf{R}\}_{s,\alpha>0} \quad (6.181)$$

that converges to a function with vanishing $L^2(\mathbf{R}^n)$ norm and non-vanishing Fourier transform

$$\lim_{s \rightarrow 0} \langle f_{s,\alpha}, f_{s,\alpha} \rangle = 0 \quad (6.182a)$$

$$\lim_{s \rightarrow 0} \int dx f_{s,\alpha}(x) = \lim_{s \rightarrow 0} f_{s,\alpha,0} = \alpha. \quad (6.182b)$$

In the case $n = 3$, an example is

$$f_{s,\alpha}(x) = \frac{\alpha s}{4\pi} \frac{e^{-s|x|}}{1 + |x|^2} \quad (6.183)$$

since⁷

$$\begin{aligned} \int dx f_{s,\alpha}(x) &= \alpha s \int_0^\infty dr \frac{r^2 e^{-sr}}{1 + r^2} \\ &= \alpha \left(1 - \frac{\pi}{2} s \cos(s) - s \sin(s) \text{Ci}(s) + s \cos(s) \text{Si}(s) \right) \xrightarrow{s \rightarrow 0} \alpha \end{aligned} \quad (6.185)$$

and

$$\int dx (f_{s,\alpha}(x))^2 = \mathcal{O}(s^2). \quad (6.186)$$

Then from (6.182a)

$$\lim_{s \rightarrow 0} \omega_F(f_{s,\alpha}, 0) = 1 = \lim_{s \rightarrow 0} \omega_F(0, f_{s,\alpha}) \quad (6.187)$$

and one can show⁸

$$\text{s-lim}_{s \rightarrow 0} U_F(f_{s,\alpha}) = U_F(0) = \mathbf{1} = V_F(0) = \text{s-lim}_{s \rightarrow 0} V_F(f_{s,\alpha}). \quad (6.188)$$

⁷Using the sine and cosine integrals

$$\text{Si}(x) = \int_0^x d\xi \frac{\sin \xi}{\xi} \quad (6.184a)$$

$$\text{Ci}(x) = - \int_x^\infty d\xi \frac{\cos \xi}{\xi} \quad (6.184b)$$

⁸Using lemma 2.3 of [AW63].

Thus from (6.182b) we find

$$\text{s-lim}_{s \rightarrow 0} U_\rho(f_{s,\alpha}) = \mathbf{1} \otimes e^{i\sqrt{2\rho\alpha}C} \quad (6.189a)$$

$$\text{s-lim}_{s \rightarrow 0} V_\rho(f_{s,\alpha}) = \mathbf{1} \otimes e^{i\sqrt{2\rho\alpha}S}. \quad (6.189b)$$

Using the double angle formulas

$$\sin(2\theta) = 2 \sin \theta \cos \theta \quad (6.190a)$$

$$\cos(2\theta) = \cos^2 \theta - \sin^2 \theta, \quad (6.190b)$$

we find that repeated application of S and C will generate $\sin(nx)$ and $\cos(nx)$ for all $n \in \mathbf{N}$. By Fourier decomposition of $L^2(S^1)$ the constant function Φ_0 is thus cyclic for S and C . Since Ω_F is cyclic for $U_F(f)$ and $V_F(g)$, we have established that Ω_ρ is cyclic for $U_\rho(f)$ and $V_\rho(g)$.

Superselection Sectors

Lecture 27: Wed, 30.01.2019

By construction, all operators of the form

$$A = \mathbf{1} \otimes T \quad (6.191)$$

where

$$[S, T] = [C, T] = 0 \quad (6.192)$$

commute with all operators from the representation $\pi_\rho(\mathcal{A})$:

$$[U_\rho(f), A] = [V_\rho(g), A] = 0. \quad (6.193)$$

Among the operators commuting with C and S are arbitrary multiplication operators

$$(T\psi)(\theta) = t(\theta) \cdot \psi(\theta). \quad (6.194)$$

Therefore the representation $(\mathcal{H}_\rho, \pi_\rho, \Omega_\rho)$ is highly reducible and the Hilbert space decomposes into superselection sectors carrying irreducible representations labelled by (improper) eigenvalues of the angle operator Θ

$$(\Theta\psi)(\theta) = \theta \cdot \psi(\theta) \quad (6.195)$$

as direct integrals

$$\mathcal{K} = \int_{S^1}^{\oplus} \frac{d\theta}{2\pi} \mathcal{K}_\theta. \quad (6.196)$$

and

$$\mathcal{H}_\rho = \mathcal{H}_F \otimes \int_{S^1}^{\oplus} \frac{d\theta}{2\pi} \mathcal{K}_\theta = \int_{S^1}^{\oplus} \frac{d\theta}{2\pi} \underbrace{(\mathcal{H}_F \otimes \mathcal{K}_\theta)}_{\mathcal{H}_{\rho,\theta}}. \quad (6.197)$$

The corresponding decomposition of the operators is

$$U_\rho(f) = \int_{S^1}^{\oplus} \frac{d\theta}{2\pi} U_{\rho,\theta}(f) \quad (6.198a)$$

$$V_\rho(g) = \int_{S^1}^{\oplus} \frac{d\theta}{2\pi} V_{\rho,\theta}(g) \quad (6.198b)$$

with

$$U_{\rho,\theta}(f) = U_F(f) \otimes e^{i\sqrt{2\rho}f_0 \cos \theta} \quad (6.199a)$$

$$V_{\rho,\theta}(g) = V_F(g) \otimes e^{i\sqrt{2\rho}g_0 \sin \theta}. \quad (6.199b)$$

Since the action on \mathcal{K}_θ is just a phase, we can equivalently write

$$U_{\rho,\theta}(f) = U_F(f) e^{i\sqrt{2\rho}f_0 \cos \theta} = \exp \left(i \left(\phi_F(f) + \sqrt{2\rho}f_0 \cos \theta \right) \right) \quad (6.200a)$$

$$V_{\rho,\theta}(g) = V_F(g) e^{i\sqrt{2\rho}g_0 \sin \theta} = \exp \left(i \left(\pi_F(g) + \sqrt{2\rho}g_0 \sin \theta \right) \right) \quad (6.200b)$$

or

$$\phi_{\rho,\theta}(f) = \phi_F(f) + \sqrt{2\rho}f_0 \cos \theta \quad (6.201a)$$

$$\pi_{\rho,\theta}(g) = \pi_F(g) + \sqrt{2\rho}g_0 \sin \theta. \quad (6.201b)$$

Formally, we find the same result as in the heuristic discussion above:

$$\phi_{\rho,\theta}(x) = \phi_F(x) + \sqrt{2\rho} \cos \theta \quad (6.202a)$$

$$\pi_{\rho,\theta}(x) = \pi_F(x) + \sqrt{2\rho} \sin \theta, \quad (6.202b)$$

i. e. the inequivalent irreducible representations are labelled by the expectation values

$$\langle \phi_{\rho,\theta}(x) \rangle = \langle \phi_{\rho,\theta} \rangle = \sqrt{2\rho} \cos \theta \quad (6.203a)$$

$$\langle \pi_{\rho,\theta}(x) \rangle = \langle \pi_{\rho,\theta} \rangle = \sqrt{2\rho} \sin \theta. \quad (6.203b)$$

As discussed in section 5.3.3 for a particle on a circle, the intertwiners relating $(\phi_{\rho,\theta}, \pi_{\rho,\theta})$ with $(\phi_{\rho,\theta'}, \pi_{\rho,\theta'})$ for $\theta \neq \theta'$ can not be represented by unitary operators, i. e. the gauge symmetry is *spontaneously broken*.

We can interpret

$$a_0 = \lim_{V \rightarrow \mathbf{R}^n} a(\chi_V) = \frac{1}{\sqrt{2\rho}} (\langle \phi_{\rho, \theta} \rangle + i \langle \pi_{\rho, \theta} \rangle) = e^{i\theta} \quad (6.204a)$$

$$a_0^* = \lim_{V \rightarrow \mathbf{R}^n} a^*(\chi_V) = \frac{1}{\sqrt{2\rho}} (\langle \phi_{\rho, \theta} \rangle - i \langle \pi_{\rho, \theta} \rangle) = e^{-i\theta} \quad (6.204b)$$

with

$$\chi_V(x) = \begin{cases} \frac{1}{\sqrt{\rho V}} & x \in V \\ 0 & x \notin V \end{cases} \quad (6.205)$$

as unitary and *commuting* annihilation and creation operators for zero momentum particles in the ground state.

6.5 The BCS Model

An important realization of the phenomena discussed for the free Bose gas in a physics model with interactions is given by the Bardeen-Cooper-Shrieffer model for superconductivity.

6.5.1 The Model

Instead of bosons, we have fermions with anti commutation relations

$$[\psi_i(f), \psi_j^*(g)]_+ := \psi_i(f)\psi_j^*(g) + \psi_j^*(g)\psi_i(f) = \delta_{ij}\langle f, g \rangle \quad (6.206a)$$

$$[\psi_i(f), \psi_j(g)]_+ = [\psi_i^*(f), \psi_j^*(g)]_+ = 0 \quad (6.206b)$$

with spin quantum numbers $i, j \in \{1, 2\}$ and complex valued square integrable functions f and g

$$\langle f, g \rangle = \int dx \bar{f}(x)g(x). \quad (6.207)$$

Again, we can interpret this as operator valued distributions

$$\psi_i(f) = \int dx \bar{f}(x)\psi_i(x) \quad (6.208a)$$

$$\psi_i^*(f) = \int dx f(x)\psi_i^*(x) \quad (6.208b)$$

with

$$[\psi_i(x), \psi_j^*(y)]_+ = \delta_{ij}\delta(x - y) \quad (6.209a)$$

$$[\psi_i(x), \psi_j(y)]_+ = [\psi_i^*(x), \psi_j^*(y)]_+ = 0. \quad (6.209b)$$

The hamiltonian in a finite volume V is

$$H_V = \int_V dx \sum_{i=1}^2 \left(\frac{1}{2m} (\nabla \psi_i^*(x)) (\nabla \psi_i(x)) - \mu \psi_i^*(x) \psi_i(x) \right) + \frac{1}{V} \int_V dx dx' dy dy' \underbrace{\psi_1^*(x) \psi_2^*(x+y) H_{\text{int.}}(y, y') \psi_2(x'+y) \psi_1(x')}_{= H_I} . \quad (6.210)$$

Despite the nonlinear interactions in H_V for $H_{\text{int.}} \neq 0$, the model can be solved exactly in the infinite volume limit [Haa62]

$$H = \lim_{V \rightarrow \mathbf{R}^n} H_V + \text{const.} \quad (6.211)$$

with the constant adjusted, such that the expectation value of the hamiltonian vanishes in the ground state. The crucial ingredient is that we will show

$$\lim_{V \rightarrow \mathbf{R}^n} [H_V, \psi^*(x)] = - \int dy \Delta(y) \psi^*(x-y),$$

i. e. that the nonlinear terms in the Heisenberg equation of motion will vanish in the infinite volume limit.

6.5.2 The Solution

Lecture 28: Fri, 01. 02. 2019

Lemma 6.15. *The fermionic annihilation and creation operators $\psi_i(f)$ and $\psi_i^*(f)$ are bounded iff f is square-integrable and*

$$\|\psi_i(f)\| = \|\psi_i^*(f)\| = \sqrt{\langle f, f \rangle}. \quad (6.212)$$

Proof. From the anti commutation relations, we find

$$\begin{aligned} (\psi_i^*(f) \psi_i(f))^2 &= \psi_i^*(f) \psi_i(f) \psi_i^*(f) \psi_i(f) \\ &= \langle f, f \rangle \psi_i^*(f) \psi_i(f) - \underbrace{\psi_i^*(f) \psi_i^*(f)}_{= 0} \underbrace{\psi_i(f) \psi_i(f)}_{= 0} \end{aligned} \quad (6.213)$$

that

$$P_i(f) = \frac{1}{\langle f, f \rangle} \psi_i^*(f) \psi_i(f) \quad (6.214)$$

is an orthogonal projection

$$P_i(f) = P_i^2(f) = P_i^*(f), \quad (6.215)$$

which is necessarily bounded

$$\|P_i(f)\| = 1. \quad (6.216)$$

Therefore

$$\|\psi_i^*(f)\| = \|\psi_i(f)\| = \sqrt{\|\psi_i^*(f)\psi_i(f)\|} = \sqrt{\langle f, f \rangle}. \quad (6.217)$$

□

Definition 6.16 (even, quasi local). A quantity Q is called *quasi-local* quantity **iff** it can be written as

$$Q(x) = \int dy_1 \dots dy_n dz_1 \dots dz_m F_{i_1, \dots, i_n; j_1, \dots, j_m}(y_1, \dots, y_n; z_1, \dots, z_m) \psi_{i_1}^*(x + y_1) \dots \psi_{i_n}^*(x + y_n) \psi_{j_1}(x + z_1) \dots \psi_{j_m}(x + z_m) \quad (6.218)$$

and F is rapidly decreasing for each $|y_i|, |z_i| \rightarrow \infty$. $Q(x)$ is called *even*, **iff** $n + m \in 2\mathbf{N}$.

We use *quantity* instead of *operator*, because it is in general an operator valued distribution.

Lemma 6.17. *If Q is an even, quasi-local quantity, its space average in the infinite volume limit*

$$\langle Q \rangle := \lim_{V \rightarrow \mathbf{R}^n} \frac{1}{V} \int_V dx Q(x) \quad (6.219)$$

commutes with all operators in the algebra generated by the annihilation and creation operators $\psi(f)$ and $\psi^(x)$.*

Proof. The essential ingredients of the proof are the rapid decrease of the function F defining Q and the boundedness of the fermionic operators. Consider, e. g.,

$$Q(x) = \int dz_1 dz_2 F(z_1, z_2) \psi_1(x + z_1) \psi_2(x + z_2) \quad (6.220)$$

and compute

$$\begin{aligned} [\langle Q \rangle, \psi_2^*(f)] &= \\ & \lim_{V \rightarrow \mathbf{R}^n} \frac{1}{V} \int_V dx \int dz_1 dz_2 F(z_1, z_2) [\psi_1(x + z_1) \psi_2(x + z_2), \psi_2^*(f)] \end{aligned}$$

$$\begin{aligned}
&= \lim_{V \rightarrow \mathbf{R}^n} \frac{1}{V} \int_V dx \int dz_1 dz_2 F(z_1, z_2) \psi_1(x + z_1) f(x + z_2) \\
&= \lim_{V \rightarrow \mathbf{R}^n} \frac{1}{V} \int_V dx \psi_1(x) \int dz_1 dz_2 F(z_1, z_2) f(x + z_2 - z_1) \\
&= \lim_{V \rightarrow \mathbf{R}^n} \frac{1}{V} \psi(g_V) \quad (6.221)
\end{aligned}$$

with⁹

$$g_V(x) = \begin{cases} \int dz_1 dz_2 F(z_1, z_2) f(x + z_2 - z_1) & \text{for } x \in V \\ 0 & \text{for } x \notin V \end{cases}. \quad (6.222)$$

Since f is square integrable and F is rapidly decreasing, g will be square integrable and remain so in the limit $V \rightarrow \mathbf{R}^n$. Therefore $\psi(g_V)$ will remain bounded in the limit $V \rightarrow \mathbf{R}^n$ and due to the factor $1/V$

$$[\langle Q \rangle, \psi_i^*(f)] = 0. \quad (6.223)$$

With the Leibniz rule for commutators, this immediately implies the generalization to arbitrary polynomials

$$[\langle Q \rangle, \psi_{i_1}^*(f_1) \cdots \psi_{i_n}^*(f_n) \psi_{j_1}(g_1) \cdots \psi_{j_m}(g_m)] = 0 \quad (6.224)$$

and one can show that it also holds for arbitrary quasi-local $Q(x)$. \square

The interaction Hamiltonian in (6.210) is *not* the spatial average of a quasi-local quantity, because there are two undamped integrations over x and x' and only a single factor of $1/V$. Nevertheless, we can compute¹⁰

⁹One could also work with

$$g(x) = \int dz_1 dz_2 F(z_1, z_2) f(x + z_2 - z_1)$$

directly to avoid discussing the translation of the finite integration volume V .

¹⁰Alternatively we can compute

$$[H_I, \psi_1(z)] = - \lim_{V \rightarrow \mathbf{R}^n} \frac{1}{V} \int_V dx' dy dy' \psi_2^*(z + y) H_{\text{int.}}(y, y') \psi_2(x' + y') \psi_1(x') \quad (6.225)$$

and from this the different types of nested commutators. For mixed annihilation and creation operators, we find from the rapid falloff of $H_{\text{int.}}$ in all directions

$$\begin{aligned}
&[[H_I, \psi_1(z)], \psi_1^*(z')] \\
&= - \lim_{V \rightarrow \mathbf{R}^n} \frac{1}{V} \underbrace{\int_V dy dy' \psi_2^*(z + y) H_{\text{int.}}(y, y') \psi_2(z' + y')}_{\text{bounded}} = 0. \quad (6.226)
\end{aligned}$$

$$\begin{aligned}
[H_I, \psi_1(z)] &= \\
\lim_{V \rightarrow \mathbf{R}^n} \frac{-1}{V} \int_V dx dx' \int dy dy' \delta(x-z) \psi_2^*(x+y) H_{\text{int.}}(y, y') \psi_2(x'+y) \psi_1(x') \\
&= \lim_{V \rightarrow \mathbf{R}^n} \frac{-1}{V} \int_V dx' \int dy dy' \psi_2^*(z+y) H_{\text{int.}}(y, y') \psi_2(x'+y) \psi_1(x') \\
&= - \lim_{V \rightarrow \mathbf{R}^n} \int dy' \underbrace{\int dy \psi_2^*(z+y) H_{\text{int.}}(y, y')}_{A_V(z, y')} \underbrace{\frac{1}{V} \int_V dx' \psi_2(x'+y) \psi_1(x')}_{\phi_V(y') \rightarrow \varphi(y') \mathbf{1}} \\
&= - \lim_{V \rightarrow \mathbf{R}^n} \int dy \psi_2^*(z+y) \underbrace{\int dy' H_{\text{int.}}(y, y') \phi_V(y')}_{\Delta_V(y) \rightarrow \Delta(y) \mathbf{1}}. \quad (6.228)
\end{aligned}$$

Similarly to proof of lemma 6.17 we can show that

$$\phi(x) = \lim_{V \rightarrow \mathbf{R}^n} \phi_V(x) = \lim_{V \rightarrow \mathbf{R}^n} \frac{1}{V} \int_V dy \psi_2(x+y) \psi_1(y) \quad (6.229)$$

commutes with all operators in the algebra generated by $\psi(f)$ and $\psi^*(f)$. Indeed

$$[\phi(x), \psi_1^*(f)] = \lim_{V \rightarrow \mathbf{R}^n} \frac{1}{V} \int_V dy \psi_2(x+y) f(y) = \lim_{V \rightarrow \mathbf{R}^n} \frac{1}{V} \underbrace{\psi_2(\alpha_{-x} f)}_{\text{bounded}} = 0 \quad (6.230)$$

where $(\alpha_x f)(y) = f(x+y)$. In an irreducible representation, $\phi(x)$ is therefore proportional to the unit for all x . Since the interaction $H_{\text{int.}}(y, y')$ is assumed to fall off rapidly for $|y|, |y'| \rightarrow \infty$, the operator

$$A_V(z, y') = \int_V dy \psi_2^*(z+y) H_{\text{int.}}(y, y') \quad (6.231)$$

Obviously, the same result follows for other spins. On the other hand, for pairs of annihilation operators, there is one undamped integration and the result doesn't vanish. However, we can use lemma 6.17 to show that the result lies in the center of the algebra and must be proportional to the unit operator in an irreducible representation

$$\begin{aligned}
[[H_I, \psi_1(z)], \psi_2(z')] &= \\
&= - \lim_{V \rightarrow \mathbf{R}^n} \frac{1}{V} \int_V dx' \underbrace{\int_V dy' H_{\text{int.}}(z'-z, y') \psi_2(x'+y') \psi_1(x')}_{\text{quasi local}} \\
&= -\Delta(z'-z) \mathbf{1}. \quad (6.227)
\end{aligned}$$

We get the same result for any pair of annihilation and creation operators with different spins, but the commutators for pairs with the same spin vanish.

is bounded for all $z, y' \in \mathbf{R}^n$ and $V \subseteq \mathbf{R}^n$. In addition it falls off rapidly in the operator norm for $|y'| \rightarrow \infty$.

In summary, we have shown that

$$[H_I, \psi_1(z)] = - \int dy \psi_2^*(z+y) \Delta(y), \quad (6.232)$$

with a *commuting*

$$\Delta(y) \mathbf{1} = \lim_{V \rightarrow \mathbf{R}^n} \frac{1}{V} \int_V dx' \int_V dy' H_{\text{int.}}(y, y') \psi_2(x'+y') \psi_1(x'). \quad (6.233)$$

This means that H_I has the same commutation relations as

$$H'_I = \int dx dy (\Delta(y) \psi_1^*(x) \psi_2^*(x+y) + \bar{\Delta}(y) \psi_2(x+y) \psi_1(x)) + \text{const.} \quad (6.234)$$

in the infinite volume limit. Since we can compute all matrix elements using the commutation relations, we can replace H_I by H'_I everywhere. We will call H' the result of substituting H'_I for H_I in H .

As we have seen in the exercises, any Hamiltonian that is a quadratic polynomial in the annihilation and creation operators can be diagonalized by a Bogolyubov transformation. Assuming that we know Δ , we obtain in momentum space

$$H' = \int dp \omega(p) (c_1^*(p) c_1(p) + c_2^*(p) c_2(p)) \quad (6.235)$$

with

$$\epsilon(p) = \frac{p^2}{2m} - \mu \quad (6.236)$$

$$\omega(p) = \sqrt{\epsilon^2(p) + |\tilde{\Delta}(p)|^2} \quad (6.237)$$

where $\tilde{\Delta}$ is the Fourier transform of Δ and the free constant has been adjusted to set the energy in the Fock ground state to zero. The new annihilation and creation operators $c_i(p)$ and $c_i^*(p)$ are related to the Fourier transforms of the original ones by

$$\tilde{\psi}_1(p) = u(p) c_1(p) - \bar{v}(p) c_2^*(-p) \quad (6.238a)$$

$$\tilde{\psi}_2(p) = \bar{v}(-p) c_1^*(-p) + u(-p) c_2(p) \quad (6.238b)$$

with

$$u(p) = \frac{\tilde{\Delta}(p)}{\sqrt{(\omega(p) - \epsilon(p))^2 + |\tilde{\Delta}(p)|^2}} \quad (6.239a)$$

$$v(p) = \frac{\omega(p) - \epsilon(p)}{\sqrt{(\omega(p) - \epsilon(p))^2 + |\tilde{\Delta}(p)|^2}}. \quad (6.239b)$$

We still need to determine Δ .

In lemma 6.17 we have shown that the space average of an even, *quasi-local* quantity is proportional the unit operator. The coefficient of the unit operator can be computed as the expectation value in the ground state Ω . However, the ground state is translation invariant and we can compute

$$\langle Q \rangle = \lim_{V \rightarrow \mathbf{R}^n} \frac{1}{V} \int_V dx (\Omega, Q(x)\Omega) = (\Omega, Q(x)\Omega). \quad (6.240)$$

This way, we obtain

$$\varphi(z) = (\Omega, \psi_2(z)\psi_1(0)\Omega) \quad (6.241)$$

and we can compute its Fourier transform using the Bogolyubov transformation (6.238)

$$\tilde{\varphi}(p) = -u(p)\bar{v}(p). \quad (6.242)$$

Given an interaction $H_{\text{int.}}$, we can compute $\tilde{\Delta}(p)$ from $\tilde{\varphi}(p)$ and insert the result in (6.239) to obtain an integral equation for φ , known as *gap equation*. For suitable realistic attractive interactions $H_{\text{int.}}$, the gap equation has not only the trivial solution $\varphi = 0$, but also a one parameter family of solutions

$$\varphi_\theta(x) = e^{i\theta}\varphi_0(x) \quad (6.243)$$

with $\theta \in [0, 2\pi)$ and $\varphi_0 : \mathbf{R}^n \rightarrow [0, \infty)$. It turns out that the nontrivial solution corresponds to a lower energy and is the interacting ground state with a condensate of *Cooper pairs*

$$\varphi(z) = (\Omega, \psi_2(z)\psi_1(0)\Omega). \quad (6.241)$$

Since there is a finite energy gap between the Fock state and the correlated ground state, Cooper pairs in the lowest energy state are not scattered and move freely at low temperatures.

6.5.3 Gauge Symmetry Breaking

Lecture 29: Wed, 06.02.2019

The classical Hamiltonian is invariant under a global gauge symmetry

$$\psi(x) \rightarrow e^{ie\alpha} \psi(x) \quad (6.244a)$$

$$\psi^*(x) \rightarrow e^{-ie\alpha} \psi^*(x) \quad (6.244b)$$

under which the condensate ϕ is *not* invariant

$$\varphi(x) \rightarrow e^{i2e\alpha} \varphi(x). \quad (6.245)$$

Using the same arguments as in the case of the free Bose gas, we see that the ground state with $\varphi \neq 0$ breaks this symmetry, because the gauge transformations can not be realized as unitary transformations.

If the system is coupled to an electromagnetic vector potential (Φ, A) by minimal coupling

$$\mu \rightarrow \mu - e\Phi(x) \quad (6.246a)$$

$$\nabla \rightarrow \nabla - ieA(x), \quad (6.246b)$$

the equations of motion are even invariant under local gauge transformations

$$\psi(x) \rightarrow e^{ie\alpha(x,t)} \psi(x) \quad (6.247a)$$

$$\psi^*(x) \rightarrow e^{-ie\alpha(x,t)} \psi^*(x) \quad (6.247b)$$

$$\Phi(x) \rightarrow \Phi(x) - e \frac{\partial \alpha}{\partial t}(x, t) \quad (6.247c)$$

$$A(x) \rightarrow A(x) - e \nabla \alpha(x, t). \quad (6.247d)$$

A non vanishing condensate of Cooper pairs also breaks this symmetry and gives a mass term to the photons, resulting in a finite penetration depth of electromagnetic fields into the superconductor, corresponding to the *Meißner effect*.

—7—

SYMMETRY AND SYMMETRY BREAKING

7.1 Wigner's Theorem

So far, we have identified unbroken symmetries of a quantum mechanical system with the existence of unitary operators in a Hilbert space representation (\mathcal{H}, π) . A priori, this is a too strong requirement, because a state of a physical system is *not* described by a single normalized vector $\Phi \in \mathcal{H}$, but by a *ray*

$$\hat{\Psi} = \{e^{i\lambda}\Psi : \lambda \in [0, 2\pi)\} \in P(\mathcal{H}) \quad (7.1)$$

and a physical symmetry is only required to preserve probabilities, i.e. the moduli of matrix elements, which obviously don't depend on the representative chosen for each ray

$$\forall \Psi \in \hat{\Psi}, \Phi \in \hat{\Phi}, \lambda, \mu \in [0, 2\pi) : |(e^{i\lambda}\Psi, e^{i\mu}\Phi)| = |(\Psi, \Phi)| = |(\hat{\Psi}, \hat{\Phi})|. \quad (7.2)$$

Definition 7.1 (Wigner Symmetry). A *Wigner symmetry* of a quantum mechanical system with states described by rays in a Hilbert space \mathcal{H} is a mapping $g : P(\mathcal{H}) \rightarrow P(\mathcal{H})$ of the projective Hilbert space of rays to itself, which preserves all transition probabilities

$$\forall \hat{\Psi}, \hat{\Phi} \in P(\mathcal{H}) : |(g\hat{\Psi}, g\hat{\Phi})| = |(\hat{\Psi}, \hat{\Phi})|. \quad (7.3)$$

However, there is a famous theorem by Wigner in Hilbert space quantum mechanics for a finite number of degrees of freedom:

Theorem 7.2 (Wigner). *Given a quantum mechanical system, that is described by the rays in a Hilbert space \mathcal{H} , all Wigner symmetries are realized by unitary or anti-unitary operators $U(g)$ with*

$$\forall \Psi \in \mathcal{H} : g\hat{\Psi} = \widehat{U(g)\Psi}, \quad (7.4)$$

where the $U(g)$ are determined upto a common phase.

An anti-unitary operator $U : \mathcal{H} \rightarrow \mathcal{H}$ satisfies

$$\forall \Psi, \Phi \in \mathcal{H} : (U\Psi, U\Phi) = \overline{(\Psi, \Phi)} = (\Phi, \Psi) \quad (7.5)$$

and is required for the realization of symmetries involving time-reversal $t \rightarrow -t$, since they switch the rôles of initial and final state in transition matrix elements.

The non-obvious aspect of Wigner's theorem is that the phases of the operators can be consistently chosen in the whole Hilbert space to obtain unitary or anti-unitary operators. In particular, it must be possible to compose symmetries without additional phases

$$U(g)U(g') = \underbrace{e^{i\phi(g,g')}}_1 U(g \circ g'). \quad (7.6)$$

Proof. See the textbook [Wei95] and the article [Bar54]. \square

7.2 Symmetry Breaking

In the case of an infinite number of **d.o.f.**, we must distinguish between symmetries realized algebraically and represented as unitary operators on Hilbert space.

Definition 7.3. An *algebraic symmetry* of a physical system is a *-automorphism or *-anti-automorphism β of the C^* -algebra \mathcal{A} generated by the observables of the system.

Definition 7.4. An *-anti-automorphism $\beta : \mathcal{A} \rightarrow \mathcal{A}$ of a C^* -algebra \mathcal{A} is an anti-linear map β preserving the structure of \mathcal{A}

$$\forall \lambda, \mu \in \mathbf{C}, A, B \in \mathcal{A} : \beta(\lambda A + \mu B) = \bar{\lambda}\beta(A) + \bar{\mu}\beta(B) \quad (7.7a)$$

$$\forall A, B \in \mathcal{A} : \beta(AB) = \beta(B)\beta(A) \quad (7.7b)$$

$$\forall A \in \mathcal{A} : \beta(A^*) = (\beta(A))^*. \quad (7.7c)$$

Definition 7.5. An *internal algebraic symmetry* is an algebraic symmetry β that commutes with all time and space translations α_t, α_x

$$\forall t \in \mathbf{R}, x \in \mathbf{R}^n : [\alpha_t, \beta] = [\alpha_x, \beta] = 0. \quad (7.8)$$

Theorem 7.6. In a “physically relevant” representation $(\mathcal{H}, \pi, \Omega)$, i. e. a representation satisfying the conditions¹ described in section 6.3.2, an internal

¹Existence of energy and momentum, stability, and existence of a ground state.

algebraic symmetry β is unbroken, i. e. represented unitarily, *iff* the ground state correlation functions are invariant under β :

$$\forall A \in \mathcal{A} : (\beta^*\omega)(A) := \omega(\beta(A)) = (\Omega, \pi(\beta(A))\Omega) = (\Omega, \pi(A)\Omega) = \omega(A) \quad (7.9)$$

Proof. If β is unbroken, it is realized by a unitary operator $U(\beta)$. Then $U(\beta)\Omega$ is also translation invariant, since β is an internal symmetry. However, since Ω is by assumption the unique translation invariant state, we must have $\beta^*\omega = \omega$. The reverse direction is corollary 4.9. \square

This theorem allows a simple characterization of symmetry breaking by

Definition 7.7 (order parameter). A ground state expectation value of an observable $A \in \mathcal{A}$ that is not invariant under an internal algebraic symmetry β

$$\omega(\beta(A)) \neq \omega(A) \quad (7.10)$$

is called an *order parameter*.

In the following section, we will for brevity leave the representation map π implicit and write $(\Omega, A\Omega)$ for $(\Omega, \pi(A)\Omega)$ since there is no possibility for confusion.

7.3 Goldstone Theorem

It turns out that the energy spectrum of systems with spontaneously broken symmetry is constrained. In fact, it appears² that the absence of **SSB** can be demonstrated by observing a gap in the excitation spectrum above the ground state:

Theorem 7.8 (Goldstone). *If*

1. $\{\beta^\lambda\}_{\lambda \in \mathbf{R}}$ is a continuous one parameter group of internal symmetries
2. β^λ is generated by a self-adjoint charge Q corresponding to a conserved current (ρ, j) , i. e.

$$\delta A := \left. \frac{d}{d\lambda} \right|_{\lambda=0} \beta^\lambda(A) = i \lim_{R \rightarrow \infty} [Q_R(t), A] \quad (7.11a)$$

$$Q_R(t) = \int_{|x| \leq R} dx \rho(x, t) \quad (7.11b)$$

²Caveat: the conditions of the following theorem are somewhat subtle and can be circumvented by important physical systems.

$$\frac{\partial}{\partial t}\rho(x, t) + \nabla j(x, t) = 0, \quad (7.11c)$$

acting on a subalgebra \mathcal{A}_0 of the algebra \mathcal{A} generated by the observables. \mathcal{A}_0 shall contain the local algebra $\mathcal{A}_L \subseteq \mathcal{A}_0$ and be stable under time evolution $\alpha_t(\mathcal{A}_0) \subseteq \mathcal{A}_0$.

3. β^λ is spontaneously broken in a cyclic representation $(\mathcal{H}, \pi, \Omega)$ with translation invariant ground state Ω , i. e. there is a self-adjoint $A \in \mathcal{A}_0$ with

$$\omega(\delta A) = i \lim_{R \rightarrow \infty} \omega([Q_R, A]) = i \lim_{R \rightarrow \infty} (\Omega, [Q_R, A]\Omega) \neq 0 \quad (7.12)$$

then, the energy spectrum at zero momentum of the states generated by the states $\{Q_R\Omega\}_{R \in \mathbf{R}}$ does not have a gap above the energy of the ground state Ω .

In other words, under the conditions of the theorem, there are excitations with arbitrarily small energy.

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Proof. In order to study the energy spectrum of a state $Q_R\Omega$ at a given momentum, 0 in the present case, in the symmetry breaking sector characterized by the order parameter $\omega(A)$, we are let to compute the Fourier transform of matrix elements like $(A\Omega, U(x)U(t)Q_R\Omega)$. Let's start with the Fourier transform of

$$\begin{aligned} J(x, t) &:= i(\Omega, [\rho(x, t), A]\Omega) = i(\Omega, \rho(x, t)A\Omega) - i(\Omega, A\rho(x, t)\Omega) \\ &= i(\Omega, \rho(x, t)A\Omega) - \overline{i(\Omega, \rho(x, t)A\Omega)} = 2 \operatorname{Im}(\Omega, \rho(x, t)A\Omega). \end{aligned} \quad (7.13)$$

Using that the automorphisms β^λ commute with the time translations

$$[\delta, \alpha_t] = \left. \frac{d}{d\lambda} \right|_{\lambda=0} [\beta^\lambda, \alpha_t] = 0 \quad (7.14)$$

and that \mathcal{A}_0 is stable under the time evolution, we can compute from the time evolution of the charge

$$Q_R(t) := U(t)Q_RU^*(t) \quad (7.15)$$

that

$$i \lim_{R \rightarrow \infty} (\Omega, [Q_R(t), A]\Omega) = i \lim_{R \rightarrow \infty} (\Omega, [U(t)Q_RU^*(t), A]\Omega)$$

$$\begin{aligned}
&= i \lim_{R \rightarrow \infty} (\Omega, U(t)[Q_R, U^*(t)AU(t)]U^*(t)\Omega) = i \lim_{R \rightarrow \infty} (\Omega, [Q_R, \alpha_{-t}(A)]\Omega) \\
&= (\Omega, \delta(\alpha_{-t}(A))\Omega) = (\Omega, \alpha_{-t}(\delta(A))\Omega) = (\Omega, \delta(A)\Omega) \\
&= i \lim_{R \rightarrow \infty} (\Omega, [Q_R, A]\Omega) =: a \neq 0. \quad (7.16)
\end{aligned}$$

If we denote by $\tilde{J}(k, t)$ the spacial Fourier transform of $J(x, t)$, we have

$$\begin{aligned}
\tilde{J}(0, t) &= \int dx J(x, t) = \lim_{R \rightarrow \infty} \int_{|x| \leq R} dx J(x, t) \\
&= \lim_{R \rightarrow \infty} \int_{|x| \leq R} dx i(\Omega, [\rho(x, t), A]\Omega) \\
&= \lim_{R \rightarrow \infty} i(\Omega, [Q_R, A]\Omega) = \lim_{R \rightarrow \infty} 2 \operatorname{Im}(\Omega, Q_R A \Omega) = a \neq 0. \quad (7.17)
\end{aligned}$$

And if we denote by $\tilde{\tilde{J}}(k, \omega)$ the double Fourier transform of $J(x, t)$, we find from the time independence of $\tilde{J}(0, t)$

$$\lim_{k \rightarrow 0} \tilde{\tilde{J}}(k, \omega) = b\delta(\omega), \quad (7.18)$$

i. e. there must be a state with zero energy in the spectral decomposition of $\operatorname{Im}(\Omega, \rho(x, t)A\Omega)$. \square

We can demonstrate this statement more explicitly by introducing a complete set of states $\{\Psi_{p,E}\}_{p \in \mathbf{R}^n, E \in \mathbf{R}}$, i. e.

$$\forall \Phi \in \mathcal{H} : \Phi = \int \frac{dp}{(2\pi)^n} \frac{dE}{2\pi} (\Psi_{p,E}, \Phi) \Psi_{p,E}, \quad (7.19)$$

with

$$P\Psi_{p,E} = p\Psi_{p,E} \quad (7.20a)$$

$$H\Psi_{p,E} = E\Psi_{p,E} \quad (7.20b)$$

or

$$U(x, t)\Psi_{p,E} = e^{-i(Et - px)}\Psi_{p,E}. \quad (7.21)$$

We can then compute, using

$$\rho(x, t) = U^*(x, t)\rho(0, 0)U(x, t), \quad (7.22)$$

in the spectral decomposition of J

$$J(x, t) = 2 \operatorname{Im}(\Omega, \rho(x, t)A\Omega)$$

$$\begin{aligned}
&= 2 \operatorname{Im} \left(\int \frac{dp}{(2\pi)^n} \frac{dE}{2\pi} (\Omega, \rho(x, t) \Psi_{p,E}) (\Psi_{p,E}, A\Omega) \right) \\
&= 2 \operatorname{Im} \left(\int \frac{dp}{(2\pi)^n} \frac{dE}{2\pi} (\Omega, U^*(x, t) \rho(0, 0) U(x, t) \Psi_{p,E}) (\Psi_{p,E}, A\Omega) \right) \\
&= 2 \operatorname{Im} \left(\int \frac{dp}{(2\pi)^n} \frac{dE}{2\pi} (\Omega, \rho(0, 0) \Psi_{p,E}) (\Psi_{p,E}, A\Omega) e^{-i(Et - px)} \right) \quad (7.23)
\end{aligned}$$

to make the x and t dependence explicit. Now the Fourier transform is trivial

$$\tilde{J}(k, \omega) = \int dx dt e^{i(\omega t - kx)} J(x, t) = 2 \operatorname{Im} ((\Omega, \rho(0, 0) \Psi_{k,\omega}) (\Psi_{k,\omega}, A\Omega)) \quad (7.24)$$

and we obtain from (7.18)

$$\lim_{k \rightarrow 0} \operatorname{Im} ((\Omega, \rho(0, 0) \Psi_{k,\omega}) (\Psi_{k,\omega}, A\Omega)) = \frac{b}{2} \delta(\omega) \neq 0. \quad (7.25)$$

This implies

$$\lim_{k \rightarrow 0} (\Omega, \rho(0, 0) \Psi_{k,\omega}) \quad (7.26)$$

must not vanish in a neighborhood of $\omega = 0$. Therefore the Goldstone boson state $\Psi_{p,E}$ must have the quantum numbers of the charge density $\rho(x, t)$.

The most important physical application of *Goldstone's Theorem* is the protection of small energy differences against perturbations.

7.3.1 A Closer Look

While the assumptions of Goldstone's theorem 7.8 are superficially innocuous, there are systems that appear to satisfy them, yet *do* exhibit an energy gap:

- the BCS-Model has a gap above the correlated ground state Ω and a U(1) gauge symmetry breaking order parameter $(\Omega, \psi(x) \psi(0) \Omega)$,
- the standard model of elementary particle physics has a $SU_L(2)$ breaking order parameter giving masses to gauge bosons and fermions, yet not massless particles with the corresponding quantum numbers.

On the other hand, Goldstone's theorem is very successful in describing a lot of physical systems that can not be handled with other tools

- among strongly interacting elementary particles, there are some that are significantly lighter than others: according to their quantum numbers, pions are composed of two quarks, protons and neutrons of three quarks

$$\begin{pmatrix} \pi^+ \\ \pi^0 \\ \pi^- \end{pmatrix} = \begin{pmatrix} u \otimes \bar{d} \\ \frac{1}{\sqrt{2}} (u \otimes \bar{u} + d \otimes \bar{d}) \\ d \otimes \bar{u} \end{pmatrix} \quad (7.27a)$$

$$\begin{pmatrix} p \\ n \end{pmatrix} = \begin{pmatrix} u \otimes u \otimes d \\ u \otimes d \otimes d \end{pmatrix}. \quad (7.27b)$$

Thus one would natively expect

$$135 \text{ GeV} \approx m_\pi \approx \frac{2}{3} m_p \approx 650 \text{ GeV}, \quad (7.28)$$

which is obviously incorrect. While we don't have the analytical tools yet to solve strongly interacting dynamics, we can use Goldstone's theorem to explain why $m_\pi \ll 2/3 \cdot m_p$. Indeed there is a $SU_L(2) \times SU_R(2)$ symmetry, that appears to be spontaneously broken to a single $SU(2)$ and Goldstone's theorem predicts a triplet of *massless Goldstone bosons* with the quantum of the broken symmetries. While perturbation theory would predict them to become heavy through strong interactions with the other particles, Goldstone's theorem predicts that they remain light,

- massless acoustic *phonons* are the Goldstone bosons resulting from the breaking of the continuous translation symmetries to the discrete lattice translations,
- *spin waves* are massless excitations in a Heisenberg ferromagnet and are the Goldstone bosons resulting from the spontaneous breaking of the rotational symmetry.

These examples suggest that Goldstone's theorem is applicable to some systems, but not to others. Therefore it is worthwhile to reexamine the conditions of the theorem

Symmetry

It is often straightforward to show that a symmetry commutes with the time evolution by studying the Lagrangian or Hamiltonian of a system.

Example 7.9. The Lagrangian

$$L = \int d^{n+1}x \left(\frac{1}{2} \frac{\partial \varphi_1}{\partial x_\mu}(x) \frac{\partial \varphi_1}{\partial x^\mu}(x) + \frac{1}{2} \frac{\partial \varphi_2}{\partial x_\mu}(x) \frac{\partial \varphi_2}{\partial x^\mu}(x) - P(\varphi_1^2(x) + \varphi_2^2(x)) \right), \quad (7.29)$$

with real valued fields $\phi_1, \phi_2 : \mathbf{R} \times \mathbf{R}^n \rightarrow \mathbf{R}$ and P an arbitrary real and even polynomial, is invariant under the transformations

$$\begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix} \rightarrow O \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix} \quad (7.30)$$

with O an arbitrary orthogonal 2×2 -matrix. *Noether's theorem* then shows that the current

$$(\rho(x), j(x)) = j_\mu(x) = \phi_2(x) \frac{\partial}{\partial x^\mu} \phi_1(x) - \phi_1(x) \frac{\partial}{\partial x^\mu} \phi_2(x) \quad (7.31)$$

is conserved in the classical theory.

However, it is *not* obvious that an invariance of the classical theory is realized in the quantum theory, even as an algebraic symmetry. We can regularize the system by putting it into a box with finite volume V and taking the limit $V \rightarrow \mathbf{R}^n$, but it is a priori not clear that the dynamics of the system respects the symmetry in this limit. Fortunately, we can prove

Theorem 7.10. *If the finite volume dynamics α_t^V converges to the infinite volume dynamics in the norm topology, then*

$$[\beta^\lambda, \alpha_t^V] = 0 \implies [\beta^\lambda, \alpha_t] = 0. \quad (7.32)$$

Proof. An automorphism of a C^* -algebra preserves the norm and is continuous in the norm topology. Then

$$\beta^\lambda(\alpha_t(A)) = \beta^\lambda \left(\underbrace{(\alpha_t - \alpha_t^V)(A)}_{\xrightarrow{V \rightarrow \mathbf{R}^n} 0} \right) + \underbrace{\beta^\lambda(\alpha_t^V(A))}_{\alpha_t^V(\beta^\lambda(A))} \rightarrow \alpha_t(\beta^\lambda(A)). \quad (7.33)$$

□

Since the finite volume dynamics α_t^V is generated by a finite volume Hamiltonian H_V , the theorem guarantees that it suffices to verify the invariance of H_V . This is usually straightforward.

Existence of a Charge

The existence of the charge Q is much more subtle. In fact, using translation invariance, it appears hopeless to define it as the integral

$$Q(t) = \lim_{R \rightarrow \infty} \int_{|x| \leq R} dx \rho(x, t)$$

of a density $\rho(x, t)$. From

$$\rho(x, t) = U(x)\rho(0, t)U^*(x) \quad (7.34)$$

and the unitarity of $U(x)$, we see that

$$\lim_{|x| \rightarrow \infty} \rho(x, t) \neq 0. \quad (7.35)$$

Fortunately, we only need the convergence of commutators

$$\forall A \in \mathcal{A} : [Q(t), A] = \lim_{R \rightarrow \infty} \int_{|x| \leq R} dx [\rho(x, t), A]$$

for the proof of Goldstone's theorem. These are much better behaved due to the locality of the elements of the algebra \mathcal{A} . But nonlocal or instantaneous interactions can violate the condition that the local algebra is stable under the time evolution. Relativistic models with finite speed of propagation are safe, unless we employ nonlocal subsidiary conditions to fix a gauge for quantizing gauge theories (e. g. Coulomb gauge).

7.3.2 *Ginzburg-Landau*

In the usual, intuitive, explanation of Goldstone's theorem, one writes an *effective potential* for the *classical* order parameter, which is minimized by a symmetry breaking values of this order parameter, as in figure 7.1. One then argues that the order parameter will settle in one of the local minima and choose the quantum fluctuations around this minimum as the effective low energy degrees of freedom. This is the essence of the *Ginzburg-Landau* theory of phase transitions and we can justify the classical treatment of the order parameter by our previous analysis, in particular the observation that the order parameter corresponds to a element of the center $Z(\mathcal{A})$ of the observable algebra. In this picture, all phases of matter, i. e. the states of macroscopic matter in the thermodynamic limit $V \rightarrow \mathbf{R}^n$ correspond to superselection sectors and are labelled by the values of the order parameters.

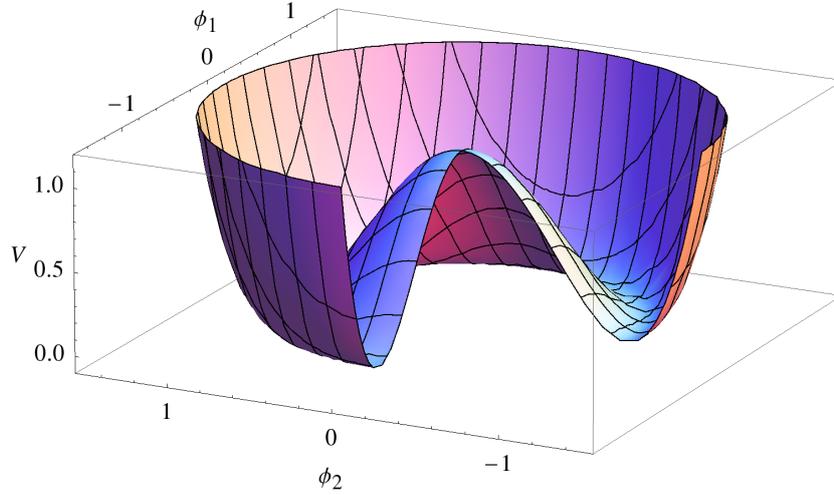


Figure 7.1: Typical “Mexican Hat” potential for the order parameter.

While this true to a large extent, recently phases have been discovered that don’t fall into this picture, cf. section 7.3.3.

Using dimensional analysis, one can derive a low energy or long distance expansion of the effective potential in the Ginzburg-Landau approach. In fact, the order parameter typically has the dimension of a positive power of an energy or inverse length, since from

$$H = \int dx \frac{1}{2m} (\nabla \psi^*(x)) (\nabla \psi(x)) + \dots \quad (7.36)$$

$\psi(x)$ must have dimension energy ^{$n/2$} or length ^{$-n/2$} . Then high powers of $\langle \psi \rangle$ are suppressed by a high inverse power of a typical energy scale Λ

$$V = \Lambda \sum_{k=0}^{\infty} C_k \left(\frac{\psi}{\Lambda^{n/2}} \right)^k \quad (7.37)$$

and it suffices to include the lowest orders, which don’t vanish for $\Lambda \rightarrow \infty$. A typical example for two real valued fields ψ_1 and ψ_2 is shown in figure 7.1

$$V(\psi_1, \psi_2) = (\psi_1^2 + \psi_2^2 - v^2)^2 = v^4 - 2v^2\psi_1^2 - 2v^2\psi_2^2 + \psi_1^4 + \psi_2^4 + 2\psi_1^2\psi_2^2. \quad (7.38)$$

This potential is invariant under rotations

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \rightarrow \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \exp \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \quad (7.39)$$

Since the order parameter lies in the center of the algebra and each value corresponds to a different inequivalent representation, the system will have to pick a value that minimizes the potential, i. e. with

$$\langle \psi_1 \rangle^2 + \langle \psi_2 \rangle^2 = v^2 . \quad (7.40)$$

By expanding around the minima, one can show that the excitations along the valley are massless, i. e. Goldstone bosons.

7.3.3 *Other Quantum Phases of Matter*

One should not forget to mention that not only symmetry breaking order parameters à la Ginzburg-Landau, as described above, characterize different phases of matter. Since the discovery of the *Quantum Hall Effect*, the concept of *Topological Order* has become very important for characterizing exotic phases of matter. Here topological invariants create elements of the center of the algebra that are robust against quantum fluctuations.

7.4 *Higgs Mechanism*

In gauge theories like quantum electrodynamics, Goldstone's theorem does not imply the existence of an observable energy gap, because the states implied by the theorem turn out to be redundant and can be removed by a suitable choice of gauge.

—A—
ACRONYMS

a.k.a. also known as

CCR Canonical Commutation Relations

CSI Cauchy-Schwarz Inequality

d.o.f. degrees of freedom

FAPP For All Practical Purposes

iff if and only if

ITP Infinite Tensor Product

LHS Left Hand Side

OTOH On The Other Hand

QM Quantum Mechanics

RHS Right Hand Side

SSB Spontaneous Symmetry Breaking

WLOG Without Loss Of Generality

wrt with respect to

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