14. Problemset "Quantum Algebra & Dynamics"
February 1, 2019

BCS / Algebraic Symmetries

14.1 Gap Equation

Derive an equation for the condensate of Cooper pairs

$$\phi(z) = (\Omega, \psi_2(z)\psi_1(0)\Omega) \tag{1}$$

assuming that the Fourier transform of $H_{\text{int.}}(y, y')$ is approximately constant and show that it has a non-trivial solution if the interaction is attractive.

14.2 Conserved Currents

Consider the BCS system described by the Hamiltonian

$$\begin{aligned} H_V = &\int_V \mathrm{d}x \sum_{i=1}^{2} \left(\frac{1}{2m} \left(\nabla\psi_i^*(x)\right)\left(\nabla\psi_i(x)\right) - \mu\psi_i^*(x)\psi_i(x) \right) \\ &+ \frac{1}{V} \int_V \mathrm{d}x\mathrm{d}x'\mathrm{d}y\mathrm{d}y'\, \psi_1^*(x)\psi_2^*(x+y)H_{\text{int.}}(y,y')\psi_2(x'+y')\psi_1(x') \end{aligned} \tag{2}$$

for *fermionic* $\{\psi_i(x)\}_{i=1,2}$ in the limit $V \to \mathbf{R}^n$.

1. H_V is obviously invariant under independent phase rotations

$$\begin{pmatrix}\psi_1(x)\\ \psi_2(x)\end{pmatrix} \to \begin{pmatrix}\mathrm{e}^{\mathrm{i}\alpha_1}\psi_1(x)\\ \mathrm{e}^{\mathrm{i}\alpha_1}\psi_2(x)\end{pmatrix} = \begin{pmatrix}\psi_1(x)\\ \psi_2(x)\end{pmatrix} + \begin{pmatrix}\delta\psi_1(x)\\ \delta\psi_2(x)\end{pmatrix} + \ldots\,, \tag{3}$$

denoted $\mathrm{U}(1) \times \mathrm{U}(1)$. Under which circumstances is it also invariant under $\mathrm{U}(2)$ transformations

$$\begin{pmatrix}\psi_1(x)\\ \psi_2(x)\end{pmatrix} \to \mathcal{U}\begin{pmatrix}\psi_1(x)\\ \psi_2(x)\end{pmatrix} = \begin{pmatrix}\psi_1(x)\\ \psi_2(x)\end{pmatrix} + \begin{pmatrix}\delta\psi_1(x)\\ \delta\psi_2(x)\end{pmatrix} + \ldots \tag{4}$$

with unitary 2×2-matrices $\mathcal{U}$?

2. Write charge operators Q_V generating the symmetry transformations

$$\delta\psi_i(x) = \mathrm{i}[Q_V, \psi_i(x)] \tag{5}$$

with

$$\delta H_V = \mathrm{i}[Q_V, H_V] = 0\,. \tag{6}$$

3. Show that

$$\frac{\mathrm{d}Q_V}{\mathrm{d}t} = 0\,. \tag{7}$$

4. Write Q_V as an integral of a charge density ρ

$$Q_V(t) = \int_V \mathrm{d}x\, \rho(x,t)\,. \tag{8}$$

5. Find a conserved current (ρ, j) containing the charge density.