13. Problemset "Quantum Algebra & Dynamics"
January 25, 2019

Free Bose Gas / The BCS Model

13.1 Finite Density

Again in the Fock representation, show that for the state with occupation number ν

$$\omega_V^{\nu/V}(f,g) := \frac{1}{\nu!}\left(\Omega_F, (a(f_V))^\nu\, U_F(f)V_F(g)\,(a^*(f_V))^\nu\,\Omega_F\right)$$
$$= \omega_F(f,g)L_\nu\left(\frac{\langle f,f_V\rangle^2+\langle g,f_V\rangle^2}{2}\right) \quad (1)$$

where L_ν is the νth Laguerre polynomial

$$L_n(x) = \sum_{k=0}^{n}\frac{(-1)^k}{k!}\binom{n}{k}x^k\,. \quad (2)$$

13.2 Occupation Number Operator

Show that

$$\mathrm{e}^{\mathrm{i}\lambda N_V}U(f) = U\left(f\cos\lambda\right)V\left(f\sin\lambda\right)\mathrm{e}^{\mathrm{i}\frac{\langle f,f\rangle}{4}\sin(2\lambda)}\mathrm{e}^{\mathrm{i}\lambda N_V} \quad (3a)$$
$$\mathrm{e}^{\mathrm{i}\lambda N_V}V(g) = V\left(g\cos\lambda\right)U\left(-g\sin\lambda\right)\mathrm{e}^{\mathrm{i}\frac{\langle g,g\rangle}{4}\sin(2\lambda)}\mathrm{e}^{\mathrm{i}\lambda N_V} \quad (3b)$$

for

$$N_V = \sum_{i\in\mathbf{N}} a^*(f_i)a(f_i) \quad (4)$$

and

$$U(f) = \mathrm{e}^{\mathrm{i}\phi(f)} \quad (5a)$$
$$V(f) = \mathrm{e}^{\mathrm{i}\pi(g)}\,. \quad (5b)$$

13.3 Bogolyubov Transform

Show that the BCS-Hamiltonian in the form

$$\begin{aligned} H = &\int \mathrm{d}x \left(\frac{1}{2m} \left(\nabla \psi^*(x)\right) \left(\nabla \psi(x)\right) - \mu \psi^*(x)\psi(x) \right) \\ &+ \int \mathrm{d}x \mathrm{d}y \, \left(\Delta(y)\psi_1^*(x)\psi_2^*(x+y) + \bar{\Delta}(y)\psi_2(x+y)\psi_1(x) \right) + \text{ const.} \end{aligned} \tag{6}$$

is brought to the form

$$H = \int \mathrm{d}p \, \omega(p) \left(c_1^*(p)c_1(p) + c_2^*(-p)c_2(-p) \right) \tag{7}$$

with

$$\epsilon(p) = \frac{p^2}{2m} - \mu \tag{8}$$

$$\omega(p) = \sqrt{\epsilon^2(p) + \left|\tilde{\Delta}(p)\right|^2} \tag{9}$$

where $\tilde{\Delta}$ is the Fourier transform of Δ, by the Bogolyubov transformation

$$\tilde{\psi}_1(p) = u(p)c_1(p) - \bar{v}(p)c_2^*(-p) \tag{10a}$$

$$\tilde{\psi}_2(p) = \bar{v}(-p)c_1^*(-p) + u(-p)c_2(p) \tag{10b}$$

with

$$u(p) = \frac{\tilde{\Delta}(p)}{\sqrt{(\omega(p) - \epsilon(p))^2 + \left|\tilde{\Delta}(p)\right|^2}} \tag{11a}$$

$$v(p) = \frac{\omega(p) - \epsilon(p)}{\sqrt{(\omega(p) - \epsilon(p))^2 + \left|\tilde{\Delta}(p)\right|^2}} \tag{11b}$$

and by ajusting the free constant to set the energy in the Fock ground state to zero.