11. Problemset "Quantum Algebra & Dynamics"
January 11, 2019

Fock Representation(s) revisited

NB: I was looking at the wrong, 12th, problemset. These problems can be solved *before* the next lecture.

11.1 Driven Oscillators

Consider this caricature

$$H = \sum_i \omega_i a_i^* a_i + g \sum_i (\bar{j}_i a_i + j_i a_i^*) \tag{1}$$

of a scalar field ϕ coupled to a classical source j

$$H = \int \mathrm{d}x \left(\frac{1}{2}\pi^2(x) + \frac{1}{2}(\nabla\phi)^2(x) + \frac{m^2}{2}\phi^2(x) + j(x)\phi(x) \right) .$$

1. Find annihilation and creation operators A_i, A_i^* and energies E_i as functions of a_i, a_i^*, ω_i, j_i and $\bar{j}_i$, such that (1) can be written

   $$H = \sum_i E_i A_i^* A_i + \text{const.} . \tag{2}$$

   Determine the shift in the ground state energy.

2. Find the unitary intertwiners U with

   $$A_i = U a_i U^* \tag{3}$$

   as a function of ω_i, j_i and $\bar{j}_i$.

3. Under which conditions on ω_i, j_i and $\bar{j}_i$ lies the interacting ground state Ω with

   $$A_i \Omega = 0 \tag{4}$$

   in the Fock representation defined by the a_i and a_i^*? Compute Ω in this case.

4. Under which conditions on ω_i, j_i and $\bar{j}_i$ are the U well defined unitary operators in the Fock representation?

11.2 Normal Ordering the Squeeze Operator

Prove that the unitary creation operators

$$U_s(\zeta) = \mathrm{e}^{\frac{1}{2}\left(\bar{\zeta}aa - \zeta a^* a^*\right)} \tag{5}$$

for squeezed states can be normal ordered as

$$U_s(\zeta) = \mathrm{e}^{-\frac{\zeta}{2}\frac{\tanh|\zeta|}{|\zeta|}a^*a^*}\mathrm{e}^{-\ln(\cosh|\zeta|)\left(a^*a+\frac{1}{2}\right)}\mathrm{e}^{\frac{\bar{\zeta}}{2}\frac{\tanh|\zeta|}{|\zeta|}aa}\,. \tag{6}$$

For convenience, introduce

$$A = \frac{1}{2}aa \tag{7a}$$

$$A^* = \frac{1}{2}a^*a^* \tag{7b}$$

$$N = a^*a + \frac{1}{2} \tag{7c}$$

to write

$$U_s(\zeta) = \mathrm{e}^{\bar{\zeta}A - \zeta A^*} = \mathrm{e}^{-\zeta\frac{\tanh|\zeta|}{|\zeta|}A^*}\mathrm{e}^{-\ln(\cosh|\zeta|)N}\mathrm{e}^{\bar{\zeta}\frac{\tanh|\zeta|}{|\zeta|}A}\,. \tag{8}$$

If the Lie algebra of the exponents in (8) closes, the Baker-Campbell-Hausdorff formula shows that we can express the exponents as linear combinations of the generators. Then it suffices to compute the coefficients in a matrix representation.

1. Show that A, A^* and N form a closed Lie algebra.

2. Show that π with

$$\pi(A) = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \tag{9a}$$

$$\pi(A^*) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{9b}$$

$$\pi(N) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{9c}$$

   is a representation of the Lie algebra. (NB: $\pi(A^*) \neq (\pi(A))^\dagger$, but we will never use that in the following!)

3. Compute the matrix exponentials in (8) to prove the normal ordering formula.