**9. Problemset "Quantum Algebra & Dynamics"**
December 14, 2018

# Coherent States / Symmetries

## 9.1 Coherent States

Consider the family $\mathcal{U}_c$ of operators

$$U_c(z) = \mathrm{e}^{za^* - \bar{z}a}\,, \tag{1}$$

where $z \in \mathbf{C}$ and

$$a^* = \frac{1}{\sqrt{2}}(x - \mathrm{i}p) \tag{2a}$$

$$a = \frac{1}{\sqrt{2}}(x + \mathrm{i}p) \tag{2b}$$

are the creation and annihilation operators for a harmonic oscillator with Hamiltonian $H = \frac{1}{2}p^2 + \frac{1}{2}x^2 = a^*a + \frac{1}{2}$.

1. Show that $U_c(z)$ creates coherent states, i. e.

$$a\,|z\rangle = z\,|z\rangle \tag{3}$$

   for $|z\rangle = U_c(z)\,|0\rangle$ with $a\,|0\rangle = 0$.

2. Show that

$$U_c(z_1)U_c(z_2) = f(z_1, z_2)U_c(g(z_1, z_2)) \tag{4}$$

   and compute $f, g : \mathbf{C}^2 \to \mathbf{C}$. Are $f$ or $g$ analytic?

3. Use the preceding result to find a homomorphism $\pi_c : \mathcal{A}_W \to \mathcal{U}_c$, i. e. a representation of the Weyl-algebra $\mathcal{A}_W$ on the standard Hilbert space of the harmonic oscillator.

## 9.2 Particle on $\mathbf{R}^3$

Consider the Weyl-Algebra $\mathcal{A}_W$ for a particle on $\mathbf{R}^3$, generated by $W(\vec{\xi},\vec{\eta})$ with $\vec{\xi},\vec{\eta}\in\mathbf{R}^3$ and

$$W(\vec{\xi}_1,\vec{\eta}_2)W(\vec{\xi}_2,\vec{\eta}_2) = \mathrm{e}^{\frac{\mathrm{i}}{2}(\vec{\xi}_1\vec{\eta}_2-\vec{\xi}_2\vec{\eta}_1)}W(\vec{\xi}_1+\vec{\xi}_2,\vec{\eta}_1+\vec{\eta}_2)\,. \tag{5}$$

1. Write down the Schrödinger representation of $\mathcal{A}_W$.

2. In classical physics we have the group of continuous symmetry transformations generated by translations $\vec{x}\mapsto\vec{x}-\vec{\alpha}$, boosts $\vec{p}\mapsto\vec{p}+\vec{\beta}$ and rotations $(\vec{x},\vec{p})\mapsto(R\vec{x},R\vec{p})$. Construct the corresponding group of automorphisms of $\mathcal{A}_W$ and determine their composition laws.

3. Consider the family $\mathcal{U}_c$ of creation operators for coherent states

$$U_c(\vec{z}) = \mathrm{e}^{\vec{z}\vec{a}^*-\bar{\vec{z}}\vec{a}}\,, \tag{6}$$

where $z\in\mathbf{C}^3$ and

$$\vec{a}^* = \frac{1}{\sqrt{2}}\,(\vec{x}-\mathrm{i}\vec{p}) \tag{7a}$$

$$\vec{a} = \frac{1}{\sqrt{2}}\,(\vec{x}+\mathrm{i}\vec{p}) \tag{7b}$$

are the creation and annihilation operators for a three dimensional harmonic oscillator and use them to realize $\mathcal{A}_W$. Construct unitary operators $U$ that realize translations, boosts and rotations via

$$\forall A\in\mathcal{A}_W : A\mapsto U^*AU\,. \tag{8}$$