6. Problemset “Quantum Algebra & Dynamics”
November 23, 2018

Direct Sum, Tensor Product & Hilbert Space(s)

6.1 Sum and Product

Consider the direct sum \( A_1 \oplus A_2 \) and tensor product \( A_1 \otimes A_2 \) of two \( C^* \)-algebras \( A_{1,2} \) and show that they can be made into \( C^* \)-algebras with the natural products and norms.

6.2 Spin Chain

Consider a chain of \( N \) spin-1/2 systems in the Hilbert space

\[
\mathcal{H}_N = \bigotimes_{i=1}^{N} \mathcal{H}^{(i)}
\]

(1)

with

\[
\forall i \in \{1, 2, \ldots, N\} : \mathcal{H}^{(i)} = \{ c_\uparrow \psi_\uparrow + c_\downarrow \psi_\downarrow : c_\uparrow, c_\downarrow \in \mathbb{C} \} \cong \mathbb{C}^2,
\]

(2)

in which the \( C^* \)-Algebra \( \mathcal{A}_N \) of observables generated by the Pauli matrices

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

(3)

is represented by

\[
\Sigma_k^{(i)} = \bigotimes_{k=1}^{i-1} 1 \otimes \sigma_k \otimes \bigotimes_{k=i+1}^{N} 1.
\]

(4)

1. Which dimension has \( \mathcal{H}_N \)?

2. Which dimension has \( \mathcal{A}_N \)?

3. Compute the commutation relations

\[
\left[ \Sigma_k^{(i)}, \Sigma_l^{(j)} \right] = \Sigma_k^{(i)} \Sigma_l^{(j)} - \Sigma_l^{(j)} \Sigma_k^{(i)}.
\]

(5)
4. Construct the states $\Psi^N_{\vec{a}} \in \mathcal{H}_N$ with the property

$$\forall i \in \{1, 2, \ldots, N\} : \left(\vec{a}^{(i)}_\Sigma \right) \Psi^N_{\vec{a}} = \Psi^N_{\vec{a}},$$

(6)

for all $\vec{a} \in \mathbb{R}^3$ with $\|\vec{a}\| = 1$.

5. Find a unitary operator $U_N(\vec{a}, \vec{b})$ with

$$U_N(\vec{a}, \vec{b}) \Psi^N_{\vec{b}} = \Psi^N_{\vec{a}}.$$

(7)

6. In the limit $N \to \infty$, we can study the Hilbert spaces

$$\mathcal{H}_{\vec{a}} = \lim_{N \to \infty} \mathcal{A}_N \Psi^N_{\vec{a}} \ni \Psi_{\vec{a}} = \lim_{N \to \infty} \Psi^N_{\vec{a}},$$

(8)

that are obtained by completing the spaces of states obtained from applying elements of $\lim_{N \to \infty} \mathcal{A}_N$ to $\Psi_{\vec{a}}$.

(a) Do we have $\Psi_{\vec{a}} \in \mathcal{H}_{\vec{b}}$?

(b) Does $U = \lim_{N \to \infty} U_N$ exist in the operator topology?