4. Problemset "Quantum Algebra & Dynamics"
November 9, 2018

More Functions / Subalgebras / Positivity

4.1 Square Root and Exponential

Use again the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1}$$

to parametrize a general complex 2×2-matrix $M \in \mathcal{M}_2$ by four complex numbers $(a_0, \vec{a})$

$$M(a_0, \vec{a}) = a_0 \mathbf{1} + \vec{a}\vec{\sigma} \,. \tag{2}$$

1. Use the holomorphic functional calculus to compute $\sqrt{M}$.
2. Use the holomorphic functional calculus to compute $\mathrm{e}^{\mathrm{i}M}$ and compare the result with the corresponding power series.

4.2 Square Root Revisited

Show $B^2 = A \geq 0$ for

$$B = \int_0^\infty \frac{\mathrm{d}x}{\pi} \frac{A}{\sqrt{\lambda}} \frac{1}{\lambda \mathbf{1} + A} \,. \tag{3}$$

by explicit calculation without using the holomorphic functional calculus.

4.3 Subalgebras

Consider a C^*-algebra $\mathcal{A}$, an element $P = P^* \in \mathcal{A}$ with

$$P^2 = P \,. \tag{4}$$

Show that the subset

$$\mathcal{A}' = P\mathcal{A}P = \{PAP : A \in \mathcal{A}\} \subseteq \mathcal{A} \tag{5}$$

is a C^*-algebra with identity $\mathbf{1}_{\mathcal{A}'} = P$.

4.4 Positivity in $*$-Algebras

Consider the vector space $\mathbf{C}^2$ as a $*$-algebra $\mathcal{A}$ with component wise addition, scalar multiplication and product

$$\alpha(x,y) + \alpha'(x',y') = (\alpha x + \alpha' x', \alpha y + \alpha' y') \tag{6a}$$
$$(x,y)(x',y') = (xx', yy') \tag{6b}$$

and involution given by

$$(x,y)^* = (\bar{y}, \bar{x})\,. \tag{6c}$$

Show that $\mathcal{A}$ is not a C^*-algebra:

1. find the positive elements $a \in \mathcal{A}$.
2. Show that the sum of two strictly positive elements is not always strictly positive.