

3. Problemset “Quantum Algebra & Dynamics”

November 2, 2018

2 × 2-Matrices

3.1 Norm

Which of the maps $\mathcal{M}_2 \rightarrow [0, \infty)$

$$M \mapsto |\det M| \quad (1a)$$

$$M \mapsto |\operatorname{tr} M| \quad (1b)$$

$$M \mapsto \sup_{ij} |M_{ij}| \quad (1c)$$

$$M \mapsto \sup_{v \in \mathbf{C}^2, \|v\|=1} \|Mv\| \quad (\text{with } \|v\| = \sqrt{|v_1|^2 + |v_2|^2}) \quad (1d)$$

define a norm on the algebra \mathcal{M}_2 of complex 2×2 -matrices? Which turn \mathcal{M}_2 into a C^* -algebra?

3.2 Spectrum and Resolvent

Use the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (2)$$

to parametrize a general complex 2×2 -matrix $M \in \mathcal{M}_2$ by four complex numbers (a_0, \vec{a}) :

$$M(a_0, \vec{a}) = a_0 \mathbf{1} + \vec{a} \vec{\sigma} = a_0 \mathbf{1} + a_1 \sigma_1 + a_2 \sigma_2 + a_3 \sigma_3 = \begin{pmatrix} a_0 + a_3 & a_1 - ia_2 \\ a_1 + ia_2 & a_0 - a_3 \end{pmatrix}. \quad (3)$$

1. For which $(a_0, \vec{a}) \in \mathbf{C}^4$ is $M(a_0, \vec{a})$

- (a) normal,
- (b) isometric,
- (c) unitary,
- (d) self-adjoint,
- (e) positive?

2. Determine the resolvent set $r_{\mathcal{M}_2}(M(a_0, \vec{a}))$ and spectrum $\sigma_{\mathcal{M}_2}(M(a_0, \vec{a}))$ for all (a_0, \vec{a}) . Handle exceptional cases.
3. Test the general results for the spectrum of normal, isometric, unitary, self-adjoint and positive matrices.
4. Compute the resolvent

$$R^{a_0, \vec{a}} : r_{\mathcal{M}_2}(M(a_0, \vec{a})) \rightarrow \mathcal{M}_2$$

$$z \mapsto (z\mathbf{1} - M(a_0, \vec{a}))^{-1} \tag{4}$$

as a 2×2 -matrix (i. e. perform the matrix inversion explicitly!).

5. Compute the projections

$$P_{\mathcal{C}}^{M(a_0, \vec{a})} = \int_{\mathcal{C}} \frac{dz}{2\pi i} R^{a_0, \vec{a}}(z) \tag{5}$$

for “interesting” \mathcal{C} explicitly. Are there qualitatively different cases to consider?