13. Problemset "Quantum Algebra & Dynamics"
January 29, 2016

The BCS Model

13.1 Bogolyubov Transform

Show that the BCS-Hamiltonian in the form

$$
\begin{aligned}
H = \int \mathrm{d}x \left(\frac{1}{2m}\left(\nabla\psi^*(x)\right)\left(\nabla\psi(x)\right) - \mu\psi^*(x)\psi(x)\right) \\
+ \int \mathrm{d}x\mathrm{d}y \left(\Delta(y)\psi_1^*(x)\psi_2^*(x+y) + \bar{\Delta}(y)\psi_2(x+y)\psi_1(x)\right) + \text{ const.}
\end{aligned} \tag{1}
$$

is brought to the form

$$
H = \int \mathrm{d}p\, \omega(p) \left(c_1^*(p)c_1(p) + c_2^*(p)c_2(p)\right) \tag{2}
$$

with

$$
\epsilon(p) = \frac{p^2}{2m} - \mu \tag{3}
$$

$$
\omega(p) = \sqrt{\epsilon^2(p) + \left|\tilde{\Delta}(p)\right|^2} \tag{4}
$$

where $\tilde{\Delta}$ is the Fourier transform of Δ, by the Bogolyubov transformation

$$
\tilde{\psi}_1(p) = u(p)c_1(p) - \bar{v}(p)c_2^*(-p) \tag{5a}
$$

$$
\tilde{\psi}_2(p) = \bar{v}(-p)c_1^*(-p) + u(-p)c_2(p) \tag{5b}
$$

with

$$
u(p) = \frac{\tilde{\Delta}(p)}{\sqrt{(\omega(p)-\epsilon(p))^2 + \left|\tilde{\Delta}(p)\right|^2}} \tag{6a}
$$

$$
v(p) = \frac{\omega(p)-\epsilon(p)}{\sqrt{(\omega(p)-\epsilon(p))^2 + \left|\tilde{\Delta}(p)\right|^2}} \tag{6b}
$$

and by ajusting the free constant to set the energy in the Fock ground state to zero.

13.2 Gap Equation*

Derive an equation for the condensate of Cooper pairs

$$\phi(z) = (\Omega, \psi_2(z)\psi_1(0)\Omega) \tag{7}$$

assuming that the Fourier transform of $H_{\text{int.}}(y, y')$ is approximately constant and show that it has a non-trivial solution if the interaction is attractive.