12. Problemset "Quantum Algebra & Dynamics"
January 15, 2015

Free Bose Gas / Squeezed States

12.1 Fock Representation

Show that

$$\omega_F(f,g) := (\Omega_F, U_F(f)V_F(g)\Omega_F) = \exp\left(-\frac{1}{4}\langle f,f\rangle - \frac{1}{4}\langle g,g\rangle - \frac{\mathrm{i}}{2}\langle f,g\rangle\right) \tag{1}$$

with

$$\langle f,g\rangle = \int \mathrm{d}x\, f(x)g(x) \tag{2}$$

in the Fock representation introduced on page 118 of the lecture notes starting with formula (6.123). See also (5.68) on page 79.

12.2 Finite Density

Again in the Fock representation, show that for the state with occupation number ν

$$\begin{aligned}\omega_V^{\nu/V}(f,g) &:= \frac{1}{\nu!}\left(\Omega_F, (a(f_V))^\nu\, U_F(f)V_F(g)\,(a^*(f_V))^\nu\,\Omega_F\right)\\ &= \omega_F(f,g)L_\nu\left(\frac{\langle f,f_V\rangle^2 + \langle g,f_V\rangle^2}{2}\right)\end{aligned} \tag{3}$$

where L_ν is the νth Laguerre polynomial

$$L_n(x) = \sum_{k=0}^{n} \frac{(-1)^k}{k!}\binom{n}{k} x^k\,. \tag{4}$$

12.3 Occupation Number Operator

Show that

$$\mathrm{e}^{\mathrm{i}\lambda N_V}U(f) = U(f\cos\lambda)\,V(f\sin\lambda)\,\mathrm{e}^{\mathrm{i}\frac{\langle f,f\rangle}{4}\sin(2\lambda)}\mathrm{e}^{\mathrm{i}\lambda N_V} \tag{5a}$$

$$\mathrm{e}^{\mathrm{i}\lambda N_V}V(g) = V(g\cos\lambda)\,U(-g\sin\lambda)\,\mathrm{e}^{\mathrm{i}\frac{\langle g,g\rangle}{4}\sin(2\lambda)}\mathrm{e}^{\mathrm{i}\lambda N_V} \tag{5b}$$

for

$$N_V = \sum_{i\in\mathbf{N}} a^*(f_i)a(f_i) \tag{6}$$

and

$$U(f) = \mathrm{e}^{\mathrm{i}\phi(f)} \tag{7a}$$

$$V(f) = \mathrm{e}^{\mathrm{i}\pi(g)}\,. \tag{7b}$$

12.4 Squeezed States revisited

Revisit the unitary creation operators

$$U_s(\zeta) = \mathrm{e}^{\frac{1}{2}\left(\bar\zeta a a - \zeta a^* a^*\right)} \tag{8}$$

for squeezed states

$$\Psi_s(\zeta) = U_s(\zeta)\Omega\,. \tag{9}$$

1. Compute the expectation values

$$\langle x\rangle = (\Psi_s(\zeta), x\Psi_s(\zeta)) \tag{10a}$$

$$\langle p\rangle = (\Psi_s(\zeta), p\Psi_s(\zeta)) \tag{10b}$$

$$\langle x^2\rangle = (\Psi_s(\zeta), x^2\Psi_s(\zeta)) \tag{10c}$$

$$\langle p^2\rangle = (\Psi_s(\zeta), p^2\Psi_s(\zeta)) \tag{10d}$$

   with

$$x = \frac{1}{\sqrt{2m\omega}}\,(a + a^*) \tag{11a}$$

$$p = \frac{1}{\mathrm{i}}\sqrt{\frac{m\omega}{2}}\,(a - a^*) \tag{11b}$$

2. Compute the uncertainties

$$(\Delta x)^2 = \left\langle (x - \langle x\rangle)^2\right\rangle \tag{12a}$$

$$(\Delta p)^2 = \left\langle (p - \langle p\rangle)^2\right\rangle \tag{12b}$$

   and their product.

3. Compute the time dependence of these expectations for the dynamics generated by the free Hamiltonian

$$H = \omega a^* a\,. \tag{13}$$