

9. Problemset “Quantum Algebra & Dynamics”

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Coherent States / Symmetries

9.1 Coherent States

Consider the family \mathcal{U}_c of operators

$$U_c(z) = e^{za^* - \bar{z}a}, \quad (1)$$

where $z \in \mathbf{C}$ and

$$a^* = \frac{1}{\sqrt{2}}(x - ip) \quad (2a)$$

$$a = \frac{1}{\sqrt{2}}(x + ip) \quad (2b)$$

are the creation and annihilation operators for a harmonic oscillator with Hamiltonian $H = \frac{1}{2}p^2 + \frac{1}{2}x^2 = a^*a + \frac{1}{2}$.

1. Show that $U_c(z)$ creates coherent states, i. e.

$$a |z\rangle = z |z\rangle \quad (3)$$

for $|z\rangle = U_c(z) |0\rangle$ with $a |0\rangle = 0$.

2. Show that

$$U_c(z_1)U_c(z_2) = f(z_1, z_2)U_c(g(z_1, z_2)) \quad (4)$$

and compute $f, g : \mathbf{C}^2 \rightarrow \mathbf{C}$. Are f or g analytic?

3. Use the preceding result to find a homomorphism $\pi_c : \mathcal{A}_W \rightarrow \mathcal{U}_c$, i. e. a representation of the Weyl-algebra \mathcal{A}_W on the standard Hilbert space of the harmonic oscillator.

9.2 Particle on \mathbf{R}^3

Consider the Weyl-Algebra \mathcal{A}_W for a particle on \mathbf{R}^3 , generated by $W(\vec{\xi}, \vec{\eta})$ with $\vec{\xi}, \vec{\eta} \in \mathbf{R}^3$ and

$$W(\vec{\xi}_1, \vec{\eta}_1)W(\vec{\xi}_2, \vec{\eta}_2) = e^{\frac{i}{2}(\vec{\xi}_1 \vec{\eta}_2 - \vec{\xi}_2 \vec{\eta}_1)} W(\vec{\xi}_1 + \vec{\xi}_2, \vec{\eta}_1 + \vec{\eta}_2). \quad (5)$$

1. Write down the Schrödinger representation of \mathcal{A}_W .
2. In classical physics we have the group of continuous symmetry transformations generated by translations $\vec{x} \mapsto \vec{x} - \vec{\alpha}$, boosts $\vec{p} \mapsto \vec{p} + \vec{\beta}$ and rotations $(\vec{x}, \vec{p}) \mapsto (R\vec{x}, R\vec{p})$. Construct the corresponding group of automorphisms of \mathcal{A}_W and determine their composition laws.
3. Consider the family \mathcal{U}_c of creation operators for coherent states

$$U_c(\vec{z}) = e^{\vec{z}\vec{a}^* - \vec{z}\vec{a}}, \quad (6)$$

where $z \in \mathbf{C}^3$ and

$$\vec{a}^* = \frac{1}{\sqrt{2}} (\vec{x} - i\vec{p}) \quad (7a)$$

$$\vec{a} = \frac{1}{\sqrt{2}} (\vec{x} + i\vec{p}) \quad (7b)$$

are the creation and annihilation operators for a three dimensional harmonic oscillator and use them to realize \mathcal{A}_W . Construct unitary operators U that realize translations, boosts and rotations via

$$\forall A \in \mathcal{A}_W : A \mapsto U^* A U. \quad (8)$$