8. Problemset "Quantum Algebra & Dynamics"

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8.1 Inclusions

Consider once more the C^*-algebra $\mathcal{M}_2$ of 2×2-Matrices $M(a_0, \vec{a})$ parametrized by four complex numbers $(a_0, \vec{a})$, using the Pauli matrices with

$$M(a_0, \vec{a}) = a_0 \mathbf{1} + \vec{a}\vec{\sigma} \,. \tag{1}$$

The states on $\mathcal{M}_2$ can be parametrized by three real numbers $\vec{\alpha}$ with $|\vec{\alpha}| \leq 1$ and

$$\omega_{\vec{\alpha}} : M \mapsto \frac{1}{2} \operatorname{tr}(M\rho(1, \vec{\alpha})) \tag{2}$$

using $\rho(\alpha_0, \vec{\alpha}) \in \mathcal{M}_2$. The pure states are those with $|\vec{\alpha}| = 1$.

Consider also the C^*-algebra $\mathcal{M}_1 \cong \mathbf{C}$ of 1×1-Matrices $m(a) = a$ and the direct sum

$$\mathcal{M}_1 \oplus \mathcal{M}_1 \subset \mathcal{M}_2 \tag{3}$$

with the inclusion maps

$$\begin{aligned} \iota_{\vec{e}} : \mathcal{M}_1 \oplus \mathcal{M}_1 &\hookrightarrow \mathcal{M}_2 \\ m(a) \oplus m(a') &\mapsto \begin{pmatrix} a & 0 \\ 0 & a' \end{pmatrix} = \frac{1}{2} M(a + a', (a - a')\vec{e}) \end{aligned} \tag{4}$$

for $|\vec{e}| = 1$.

NB: iff the general formulae become too messy for your taste, you can restrict yourself in the following to the special case $\vec{e} = (0, 0, 1)$.

1. Show that $\iota_{\vec{e}}$ is a C^*-algebra homomorphism.
2. Show that $\omega_{\vec{e},\vec{\alpha}} = \omega_{\vec{\alpha}} \circ \iota_{\vec{e}}$ is a state on $\mathcal{M}_1 \oplus \mathcal{M}_1$. Give a concrete representation in the chosen basis.
3. Give necessary and sufficient conditions on $\vec{e}$ and $\vec{\alpha}$ for $\omega_{\vec{e},\vec{\alpha}}$ to be pure.
4. Construct the cyclic representations $(\mathcal{H}_{\vec{e},\vec{\alpha}}, \pi_{\vec{e},\vec{\alpha}}, \Omega_{\vec{e},\vec{\alpha}})$ of $\mathcal{M}_1 \oplus \mathcal{M}_1$ from $\omega_{\vec{e},\vec{\alpha}}$ in the special cases

$$\vec{\alpha} = (0, 0, 0) \tag{5a}$$
$$\vec{\alpha} = (0, 0, 1) \,. \tag{5b}$$

5. Give the concrete matrix realizations of

$$\pi_{\vec{e},\vec{\alpha}}(m(a) \oplus m(a'))\,. \tag{6}$$

6. Which representations are irreducible? Determine the invariant subspace(s) for the others.

7. Verify that

$$\pi_{\vec{e},\vec{\alpha}} = \pi_{\vec{\alpha}} \circ \iota_{\vec{e}} \tag{7}$$

with $\pi_{\vec{\alpha}}$ from problem 7.1.

8.2 Circle

Consider the algebra $C(S^1)$ of bounded complex valued continuous functions $f : S^1 \to \mathbf{C}$ on the unit circle. Perform the GNS construction of a representation for the following linear functionals $\omega : C(S^1) \to \mathbf{C}$:

$$f \mapsto f(0) \tag{8a}$$

$$f \mapsto \int_0^{2\pi} \frac{\mathrm{d}\phi}{2\pi} f(\phi) \tag{8b}$$

8.3 Gaussian Integrals

Starting from

$$P_\pi = \int \frac{\mathrm{d}\xi \mathrm{d}\eta}{2\pi} \mathrm{e}^{-\frac{\xi^2+\eta^2}{4}} \pi(W(\xi,\eta)) \tag{9}$$

and

$$W(\xi,\eta)W(\xi',\eta') = \mathrm{e}^{\frac{\mathrm{i}}{2}(\xi\eta'-\eta\xi')} W(\xi+\xi',\eta+\eta') \tag{10}$$

show that

$$P_\pi \pi(W(\xi,\eta)) P_\pi = \mathrm{e}^{-\frac{\xi^2+\eta^2}{4}} P_\pi \tag{11}$$

by a suitable variable transformation and evaluating two (identical) Gaussian integrals.