7. Problemset "Quantum Algebra & Dynamics"
November 27, 2015

GNS Construction

7.1 Spins

Consider again the C^*-algebra $\mathcal{M}_2$ of 2×2-Matrices $M(a_0, \vec{a})$ parametrized by four complex numbers $(a_0, \vec{a})$, using the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1}$$

with

$$M(a_0, \vec{a}) = a_0 \mathbf{1} + \vec{a}\vec{\sigma}. \tag{2}$$

As you have shown in problem 5.1, the states on $\mathcal{M}_2$ can be parametrized by three real numbers $\vec{\alpha}$ with $|\vec{\alpha}| \leq 1$ and

$$\begin{aligned} \omega_{\vec{\alpha}} : \mathcal{M}_2 &\to \mathbf{C} \\ M &\mapsto \frac{1}{2} \operatorname{tr}(M\rho(1, \vec{\alpha})) \end{aligned} \tag{3}$$

using $\rho(\alpha_0, \vec{\alpha}) \in \mathcal{M}_2$. The pure states are those with $|\vec{\alpha}| = 1$.

1. Perform the GNS construction of cyclic representations $(\mathcal{H}_{\vec{\alpha}}, \pi_{\vec{\alpha}}, \Omega_{\vec{\alpha}}) = (\mathcal{H}_{\omega_{\vec{\alpha}}}, \pi_{\omega_{\vec{\alpha}}}, \Omega_{\omega_{\vec{\alpha}}})$ in the special cases

$$\vec{\alpha} = (0, 0, 0) \tag{4a}$$
$$\vec{\alpha} = (0, 0, a) \qquad (\text{with } |a| < 1) \tag{4b}$$
$$\vec{\alpha} = (0, 0, 1)\ . \tag{4c}$$

2. Give the concrete matrix realizations of

$$\pi_{\vec{\alpha}}(M(a_0, \vec{a}))\,. \tag{5}$$

3. Show that (4c) leads to an irreducible and (4a) and (4b) to reducible representations. Determine the invariant subspace(s) in the latter cases.

4. Try to repeat the above problems in the general case

$$|\vec{\alpha}| < 1 \tag{6a}$$
$$|\vec{\alpha}| = 1\,. \tag{6b}$$