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2. Problemset "Quantum Algebra & Dynamics"
October 23, 2015

Adjoined Units, Ideals and Factor Algebras

2.1 Adjoining a Unit

Just like in Theorem 2.1, let $\mathcal{A}$ be a C^*-algebra *without* identity and $\bar{\mathcal{A}}$ denote the set of pairs

$$\bar{\mathcal{A}} = \{(\alpha, A) : \alpha \in \mathbf{C}, A \in \mathcal{A}\} . \tag{1}$$

The $*$-algebra operations are again

$$\mu(\alpha, A) + \lambda(\beta, B) = (\mu\alpha + \lambda\beta, \mu A + \lambda B) \tag{2a}$$
$$(\alpha, A)(\beta, B) = (\alpha\beta, \alpha B + \beta A + AB) \tag{2b}$$
$$(\alpha, A)^* = (\bar{\alpha}, A^*) \tag{2c}$$

We can define a norm via

$$\|(\alpha, A)\|_{\bar{\mathcal{A}}} = \sup_{B \in \mathcal{A}, \|B\|=1} \|\alpha B + AB\|_{\mathcal{A}} . \tag{3}$$

1. Show that (3) satisfies the triangle inequality

$$\|(\alpha, A) + (\beta, B)\|_{\bar{\mathcal{A}}} \le \|(\alpha, A)\|_{\bar{\mathcal{A}}} + \|(\beta, B)\|_{\bar{\mathcal{A}}} . \tag{4}$$

2. Show that (3) satisfies the product inequality

$$\|(\alpha, A)(\beta, B)\|_{\bar{\mathcal{A}}} \le \|(\alpha, A)\|_{\bar{\mathcal{A}}} \|(\beta, B)\|_{\bar{\mathcal{A}}} . \tag{5}$$

2.2 Ideals

A subspace $\mathcal{B} \subseteq \mathcal{A}$ is called a left ideal, if $\forall A \in \mathcal{A}, B \in \mathcal{B} : AB \in \mathcal{B}$. A subspace $\mathcal{B} \subseteq \mathcal{A}$ is called a right ideal, if $\forall A \in \mathcal{A}, B \in \mathcal{B} : BA \in \mathcal{B}$. If $\mathcal{B}$ is both a left and a right ideal it is called a two sided ideal.

1. Show that every ideal is a (sub-)algebra.

2. Show that if $\mathcal{B}$ is self adjoint and a left or right ideal, it is necessarily two sided.

2.3 Factor Algebras

Let $\mathcal{I}$ be a two sided ideal of an algebra $\mathcal{A}$.

1. Show that the factor space $\mathcal{A}/\mathcal{I}$ is also an algebra, i. e. that the algebra operations are well defined for the equivalence classes

$$[A] = \{A + I : I \in \mathcal{I}\}. \tag{6}$$

2. Show that this is also true for $\mathcal{A}/\mathcal{I}$ if $\mathcal{A}$ is a

   (a) $*$-algebra and $\mathcal{I} = \mathcal{I}^*$
   (b) Banach algebras and $\mathcal{I}$ is complete.

Caveat: a two sided ideal in a $*$-algebra is *not* necessarily self adjoint. This can be seen from Stefan Waldmann's "universal counterexample" of $\mathcal{A} = C(S^2)$ with $(fg)(x) = f(x)g(x)$ and $f^*(x) = \bar{f}(-x)$. Then $\mathcal{I} = \{f \in \mathcal{A} : f(\text{north pole}) = 0\}$ is obviously a two sided ideal, but $\mathcal{I}^* = \{f \in \mathcal{A} : f(\text{south pole}) = 0\} \neq \mathcal{I}$.