1. Problemset "Quantum Algebra & Dynamics"
October 16, 2015

Unbounded Operators / Dynamics of States

1.1 CCRs vs. Boundedness

Consider two bounded operators A and B on a Hilbert space $\mathcal{H}$, i. e.

$$\exists c_A \in \mathbf{R} : \forall \psi \in \mathcal{H} : \|A\psi\| \leq c_A \|\psi\| \tag{1a}$$
$$\exists c_B \in \mathbf{R} : \forall \psi \in \mathcal{H} : \|B\psi\| \leq c_B \|\psi\| \,. \tag{1b}$$

Show that the canonical commutation relations

$$[A, B] = AB - BA = \mathrm{i} \tag{2}$$

are inconsistent with the assumption of boundedness for the operators A and B.

NB: it is *not* necessary to find an original proof. It suffices to find, understand and present a proof from the literature.

1.2 Classical Dynamics on the 2-Torus

Consider a classical dynamical system with the 2-Torus $T^2 = S^1 \times S^1$ as phase space Γ (this is not a cotangent bundle, but it has the technical advantage of being compact).

Using standard coordinates $(\theta_1, \theta_2) \in [0, 2\pi)^2$, a consistent Poisson bracket is given by

$$\{f, g\} = \left(\frac{\partial f}{\partial \theta_1} \frac{\partial g}{\partial \theta_2} - \frac{\partial f}{\partial \theta_2} \frac{\partial g}{\partial \theta_1} \right) \,. \tag{3}$$

Assume that the Hamiltonian is

$$\begin{aligned} H &: \Gamma \to \mathbf{R} \\ (\theta_1, \theta_2) &\mapsto H(\theta_1, \theta_2) = c \cos \theta_1 \end{aligned} \,. \tag{4}$$

In order to be well defined globally, the Hamiltonian must be periodic in θ_1 and θ_2. This is the simplest choice.

1. Derive the equations of motion

2. Determine the flow Φ of a phase space point $(\theta_1, \theta_2) \in \Gamma$

3. Determine the time evolution of the state ω, where

$$\omega(f) = \int_\Gamma \mathrm{d}^2\theta\, \omega(\theta) f(\theta) \tag{5}$$

with

$$\begin{aligned} \omega &: \Gamma \to \mathbf{R} \\ (\theta_1, \theta_2) &\mapsto \omega(\theta_1, \theta_2) = \frac{1}{\pi^2} \sin^2\theta_1 \, \sin^2\theta_2 \end{aligned} . \tag{6}$$

1.3 Classical Dynamics on the 2-Sphere

Consider a classical dynamical system with the 2-Sphere S^2 as phase space Γ (this is again not a cotangent bundle, but it has the technical advantage of being compact and is highly symmetric).

Using standard spherical coordinates $(\theta, \phi) \in [0, \pi) \times [0, 2\pi]$, a consistent Poisson bracket is given by

$$\{f, g\} = \frac{1}{\sin\theta} \left(\frac{\partial f}{\partial \theta} \frac{\partial g}{\partial \phi} - \frac{\partial f}{\partial \phi} \frac{\partial g}{\partial \theta} \right) \tag{7}$$

Assume that the Hamiltonian is

$$\begin{aligned} H &: \Gamma \to \mathbf{R} \\ (\theta, \phi) &\mapsto H(\theta, \phi) = c \cos\theta \end{aligned} . \tag{8}$$

In order to be well defined globally, the Hamiltonian must be periodic in θ and ϕ. This one of the simplest choices.

1. Show that the Poisson bracket satisfies all requirements.

2. Determine the flow Φ of a phase space point $(\theta, \phi) \in \Gamma$

3. Determine the time evolution of the state ω, where

$$\omega(f) = \int_\Gamma \sin\theta \, \mathrm{d}\theta \mathrm{d}\phi \, \omega(\theta, \phi) f(\theta, \phi) \tag{9}$$

with

$$\begin{aligned} \omega &: \Gamma \to \mathbf{R} \\ (\theta, \phi) &\mapsto \omega(\theta, \phi) = \frac{2}{\pi^2} \sin\theta \, \cos^2\phi \end{aligned} . \tag{10}$$