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Non-Abelian statistics and a hierarchy of fractional spin liquids in spin-S antiferromagnets

Burkhard Scharfenberger,[1] Ronny Thomale,[2] and Martin Greiter[1]

[1]*Institut für Theorie der Kondensierten Materie and DFG Center for Functional Nanostructures (CFN), Karlsruhe Institute of Technology (KIT), Campus Süd, D-76128 Karlsruhe, Germany*

[2]*Department of Physics, Princeton University, Princeton, New Jersey 08544, USA*

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We propose a hierarchy of spin liquids with spin S, which are generated via a Schwinger boson projection from $2S$ (Abelian) chiral spin liquids with $S = 1/2$. We argue that the P and T invariant liquids we construct for integer spin S support SU(2) level $k = S$ non-Abelian statistics. We conjecture that these spin liquids serve as paradigms for antiferromagnets which are disordered through mobile charge carriers.

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Introduction. Topological phases have become a major branch of research in condensed-matter physics.[1] Intriguing features of this kind of order, which is not characterized by a broken symmetry (and hence a local order parameter), but depends on global properties of the system, have been investigated in a plethora of different models. A fundamental understanding of a topological phase has been accomplished in the quantum Hall effect.[2] There, the formation of quantized plateaus in the off-diagonal conductivity is connected to the manifestation of gapless edge states in a bulk-gapped system which are robust against local perturbations of the two-dimensional electron gas. Many areas where topological phases have been discovered are intimately linked to the quantum Hall effect. Kane and Mele[3] recently opened up the field of topological insulators in two spatial dimensions, when they generalized a graphene quantum Hall model due to Haldane[4] by introducing spin and a spin-orbit coupling term which exhibits a quantum spin Hall effect with nonchiral edge states.

The concept of fractional quasiparticle statistics[5,6] as another possible manifestation of topological order has been formulated in the context of the fractional quantum Hall (FQH) effect.[7,8] In addition to Abelian statistics, where the many-particle wave function acquires a fractional phase upon braiding of quasiparticles,[5] certain systems exhibit non-Abelian fractional statistics, where the braiding induces rotations in the topologically degenerate ground-state manifold.[6] The paradigm for a state with non-Abelian statistics is the Pfaffian state,[9,10] which has been proposed to describe the FQH plateau at $\nu = 5/2$.[10] The quasihole excitations of this state directly relate to topologically protected vortex core zero modes of certain charged and neutral superfluids, such as chiral superconductors and superfluid ^{3}He.[11,12]

Spin liquids are another diverse area where topological phases appear.[13] There, quantum fluctuations manage to suppress any formation of magnetic or other kind of local (e.g., spin solid) order which would break the continuous symmetries, such as global SU(2) spin or the translational symmetry, of the system.[14] Spin liquids have attracted particular attention in the context of high-T_c superconductivity,[15] and are now established as major players in the development of field theory and quantum criticality.[16] Whereas previously mentioned examples such as the quantum Hall effect or topological insulators generally promise a rather accessible detection of topological order through the classification of the edge states, the situation is more subtle for spin liquids. While they do not exhibit gapless edge states in general, spin liquids still possess a hidden topological order that is emergent through the fractionalization of quantum numbers and quasiparticle statistics. In fact, even before the FQH effect was discovered, fractional quantum numbers had been observed in polyacetylene, where the elementary spin excitations were found to carry not $S = 1$, but $S = 1/2$.[17] One particular analytically well-defined approach to the field of spin liquids is motivated by the quantum Hall effect. Based on an idea by Lee, Kalmeyer and Laughlin[18] proposed and developed the (Abelian) chiral spin liquid (CSL),[19] which in essence is a bosonic quantum Hall wave function of spin-flip operators. The elementary excitations are spinons, which carry $S = 1/2$, no charge, and obey half-Fermi statistics, i.e., they accumulate an Abelian phase of $\pi/2$ upon braiding. While the CSL did not turn out to be directly relevant to high-T_c superconductors, it is the system in which Wen[1] established the basic notion of topological order by computing its topological degeneracy. The low-energy physics is accordingly described by a Chern-Simons theory. Further independent branches diversified the field of spin liquids, such as SU(N) CSLs potentially realized in ultracold atomic gases,[20] and the Z_2 spin liquids which, in addition to spinons, possess vison excitations which carry no spin but couple to an emergent Z_2 gauge field. Z_2 liquids appear to be good candidates for certain frustrated $S = 1/2$ antiferromagnets and dimer models,[21,22] and further describe some prominent models in which topological phenomena can be discussed on an analytical footing. These models include Kitaev's toric code and honeycomb models.[23]

In this Rapid Communication, we use the CSL as a building block to generate a hierarchy of SU(2) invariant quantum spin liquids as candidate states for disordered spin S antiferromagnets. The reason there is no order in these systems could be frustration, but we believe that our constructions are more likely to be realized in antiferromagnets with itinerant holes, as the holons one would obtain in the liquids are likely to be vastly more mobile than in competing models. Relating CSLs to a quantum antiferromagnet, the ostensible drawback is that, as opposed to the Z_2 spin liquids, the CSL breaks parity (P) and time reversal (T), while antiferromagnets generically conserve these symmetries on a Hamiltonian level and should only exhibit spontaneous breaking of P and T in very peculiar cases. For integer spin S, this obstacle can be circumvented by combining a total of $2S$ CSLs with opposite chiralities via a Schwinger boson projection technique.[24] The resulting states hence retain the topological features encoded in the CSL *and* restore the symmetries P and T, establishing a promising route to define spin-liquid candidate states for disordered quantum antiferromagnets. As one particularly interesting example, we propose and investigate an $S = 2$ liquid which conserves P and T and exhibits non-Abelian spinon excitations, and so opens up another perspective on nontrivial topological order in higher spin antiferromagnets.

Chiral spin liquid. Let us define the fundamental building block of our spin-liquid hierarchy construction. Consider a rhombic $S = 1/2$ lattice in the complex plane $\eta_{n,m} = na + mb$, where $n,m \in \mathbb{Z}$, $a,b \in \mathbb{C}$, and $\mathrm{Im}\, b\bar{a} = 1$, where Im denotes the imaginary part. The CSL state for a circular droplet with N sites is given by

$$|\Psi_+^{\mathrm{CSL}}\rangle = \sum_{\{z_i\}} \psi_+^{\mathrm{CSL}}(z_1,\ldots,z_M) S_{z_1}^+ \cdots S_{z_M}^+ |\downarrow\rangle_N, \qquad (1)$$

where we use complex coordinates z_i to denote the positions of the $M = N/2$ spin-flip operators on the lattice. $|\downarrow\rangle_N$ is the fully spin-down polarized state, i.e., $|\downarrow\rangle_N = |\downarrow \cdots \downarrow\rangle$, and ψ_+^{CSL} is the CSL wave function

$$\psi_+^{\mathrm{CSL}}(z_1,\ldots,z_M) = \prod_{j<k}^{M}(z_j - z_k)^2 \prod_{j=1}^{M} G(z_j) e^{-\frac{\pi}{2}|z_j|^2}. \qquad (2)$$

Here $G(\eta_{n,m}) = (-1)^{(n+1)(m+1)}$ is a gauge factor which is valued -1 on the doubled unit-cell superlattice and 1 otherwise and is needed to assure the singlet property of the CSL. Up to $G(z_j)$, the form invariance to a bosonic FQH Laughlin droplet wave function at a Landau-level filling fraction $\nu = 1/2$ is apparent, i.e., the CSL can be interpreted as bosons at half filling in a fictitious magnetic field of one Dirac quantum per plaquet. The CSL possesses chirality "+," i.e., $\langle\chi\rangle = \langle \boldsymbol{S}_i(\boldsymbol{S}_j \times \boldsymbol{S}_k)\rangle > 0$, where the sites i,j,k span a triangle which is oriented such that $\mathrm{Im}\{(\eta_k - \eta_i)/(\eta_j - \eta_i)\} > 0$. As $\chi \to -\chi$ under P and T, the CSL (2) breaks P and T. A CSL of opposite chirality "−" is obtained by complex conjugation $z \to \bar{z}$. The spinon excitations are the analog of the quasiholes in the quantized Hall states, and obey half-fermion statistics $\theta = \pm\pi/2$, with the sign depending on the chirality. This also applies to holon excitations when the CSL itself is studied away from half filling of electrons. The CSL can be generated alternatively through Gutzwiller projection of filled Landau levels for $\uparrow$ and for $\downarrow$ spins, and hence be defined on any lattice type.[24–26]

Topological degeneracy. Aside from the fractional statistics of the spinons, the nontrivial topology of the CSL manifests itself in a degeneracy on a nontrivial manifold such as the torus, which we obtain by imposing periodic boundary conditions. There are various ways to determine this degeneracy for CSLs. Given any ground-state wave function, one way would be to obtain the topological entanglement entropy.[27,28] With the computational facilities presently available, however, we found it numerically challenging to write out the state vectors of hierarchy states with higher spin S for sufficiently large clusters. A more viable approach is to exploit the closed analytic expression (2) of the CSL ground-state wave function. For the torus, topological degeneracies (TDs) arise as we demand quasiperiodicity under magnetic translations of a single particle along one of the principal directions $\hat{1}$ and $\hat{\tau}$, which span the (parallelogram-shaped) principal region $\mathscr{P}_\tau = \{u + v\tau : u,v \in [0,1[\subset \mathbb{R}, \mathrm{Im}\,\tau > 0\}$. To formulate the CSL on the torus,[29] we replace the factors $(z_i - z_j)^2$ by $\vartheta_{--}^2(z_i - z_j|\tau)$, where ϑ_{--} is the odd Jacobi theta function.[30] Demanding that the wave function is strictly invariant under translations of lattice sites by 1 and τ, one finds that, in order to cancel any position dependence in the phase factors, one has to introduce a set of two parameters stemming from the power of the Jastrow factor, the center-of-mass zeros Z_ν. Their number, 2, then corresponds to the TD of the "toroidalized" state and is the ground-state degeneracy of a Hamiltonian with a suitable potential to stabilize the CSL on a torus geometry.[31] Numerically, when we write down the CSL wave functions for different values of the parameters Z_ν, $\nu = 1,2$, they will span a two-dimensional functional space

$$\dim\{|\psi_\pm^{\mathrm{CSL}}\{Z_\nu\}\rangle : Z_\nu \in \mathscr{P}_\tau\} = 2. \qquad (3)$$

This is the scheme we will use to determine the TD of all hierarchy states below.

Schwinger boson projection. In order to construct the spin-liquid hierarchy, we first rewrite the CSL wave function in terms of Schwinger bosons:[24,32]

$$|\psi_+^{\mathrm{CSL}}\rangle = \hat{\Psi}_+^{\mathrm{CSL}}[a^\dagger, b^\dagger]|0\rangle, \qquad (4)$$

where

$$\hat{\Psi}_+^{\mathrm{CSL}}[a^\dagger, b^\dagger] = \sum_{\{z_i;w_j\}} \psi_+^{\mathrm{CSL}}(z_1,\ldots,z_M) a_{z_1}^\dagger \cdots a_{z_M}^\dagger b_{w_1}^\dagger \cdots b_{w_M}^\dagger, \qquad (5)$$

and the sum extends over all possible distributions of disjoint sets of sites $\{z_1,\ldots,z_M\}$ and $\{w_1,\ldots,w_M\}$ on the lattice. The operators $a_i^\dagger$ and $b_j^\dagger$ create auxiliary bosons, with the correspondence between a spin S,S^z state and a state with $2S$ Schwinger bosons is given by

$$|S,S^z\rangle = \frac{(a^\dagger)^{S+S^z}(b^\dagger)^{S-S^z}}{\sqrt{(S+S^z)!(S-S^z)!}}|0\rangle. \qquad (6)$$

$S = 1$ liquids. Let us first study the case of constructing $S = 1$ spin liquids from the projection of two CSLs. In the Schwinger boson basis, we can accomplish the projection of the $S = 1/2$ CSLs by simply multiplying their creation operators, as illustrated in Fig. 1. Since the operators $a_i^\dagger, b_j^\dagger$ are bosonic, no reordering is needed. When we merge two CSLs, the resulting on-site terms of the form $(a_i^\dagger)^2, a_i^\dagger b_i^\dagger$ or $(b_i^\dagger)^2$ inherently belong to the symmetric $S = 1$ part of the on-site product space $\frac{1}{2} \otimes \frac{1}{2}$. We can build two different $S = 1$ liquids out of two $S = 1/2$ CSLs, depending on whether they have equal or opposite chirality. In the first case, we obtain the $S = 1$ non-Abelian chiral spin liquid (NACSL)[32]

$$|\psi_+^{\mathrm{NACSL}}\rangle = (\hat{\Psi}_+^{\mathrm{CSL}}[a^\dagger, b^\dagger])^2|0\rangle. \qquad (7)$$

The NACSL still breaks P and T, but exhibits (Ising-type) non-Abelian spinon excitations, as it resembles the bosonic Pfaffian state at Landau-level filling $\nu = 1$. To obtain its TD,

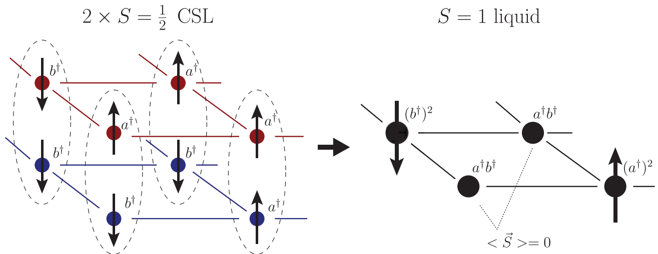

FIG. 1. (Color online) We create a spin $S = 1$ liquid out of two $S = 1/2$ CSLs by projection onto the symmetric spin configuration on each site. Schwinger bosons ensure this is done automatically when multiplying the liquids.

we create different liquids parametrized by the CSL center-of-mass zeros before projection and compute the dimension of the functional space. As listed in Table I, the NACSL has a TD of 3. Since the TD is lower than 4, i.e., lower than the product of the two constituent CSL liquids, there is a blocking mechanism, indicating that the statistics is non-Abelian.[33]

Projecting two CSLs of opposite chirality gives the spin $S = 1$ chirality liquid (S1CL),[24]

$$|\psi^{\rm S1CL}\rangle = \hat{\Psi}_+^{\rm CSL}[a^\dagger, b^\dagger]\hat{\Psi}_-^{\rm CSL}[a^\dagger, b^\dagger]|0\rangle. \quad (8)$$

The resulting wave function is real, implying the absence of a net chirality and thus invariance under the action of P and T. We obtain a TD of 4. The absence of blocking suggests that the spinons keep their statistical properties from the CSLs, but now carry a chirality quantum number. Due to the invariance under P and T, this spin liquid is an interesting trial state for $S = 1$ antiferromagnets which can be defined on any lattice.

Spin-liquid hierarchy. We now generalize this construction to arbitrary spin S, i.e., to a hierarchy of states obtained by projecting an arbitrary number of CSLs. Specifically, we form spin $S = k/2$, $k \in \mathbb{N}$ liquids from k copies of the CSL as the basic building blocks. If we constrain ourselves to just one type of chirality, the wave functions of the spin S liquids

$$\psi_+^{k-\rm CSL} = (\hat{\Psi}_+^{\rm CSL}[a^\dagger, b^\dagger])^k|0\rangle \quad (9)$$

assume the form of bosonic Read-Rezayi quantum Hall states at Landau-level filling fractions $\nu = k/2$.[34]

TABLE I. Topological degeneracies a different hierarchy liquids. The liquids have been investigated numerically up to 16 sites on a square lattice by varying the center-of-mass zeros to probe the space of degenerate states.

Spin	Liquid	Topological degeneracy
1/2	CSL	2
1	$(\mathrm{CSL}_+)^2$	3
	$\mathrm{CSL}_+\mathrm{CSL}_-$	4
3/2	$(\mathrm{CSL}_+)^3$	4
	$(\mathrm{CSL}_+)^2\mathrm{CSL}_-$	6
2	$(\mathrm{CSL}_+)^4$	5
	$(\mathrm{CSL}_+)^2(\mathrm{CSL}_-)^2$	9
$k/2$	$(\mathrm{CSL}_+)^k$	$k+1$
k	$(\mathrm{CSL}_+)^k(\mathrm{CSL}_-)^k$	$(k+1)^2$

Using conformal field theory, one finds that the TD of the kth Read-Rezayi states is given by $k + 1$,[35] which matches the results we obtain numerically (see Table I). For $S = 3/2$ or any other half-integer spin, we cannot form a P and T invariant liquid. Taking two CSLs of one chirality and one CSL of the other chirality gives a liquid with a sixfold TD. The construction suggests that this liquid supports spinons with (Ising-type) non-Abelian statistics in the chiral sector where two CSLs have been projected together and spinons with Abelian half-Fermi statistics in the other chirality sector.

$S = 2$ *chirality liquid.* A particularly interesting spin-liquid trial state can be constructed for $S = 2$. Taking two CSLs of each chirality before projection, we obtain a P and T invariant liquid. We construct the state either from two S1CLs, both of which support Abelian spinons and are P and T invariant, or from two NACSLs with (Ising-type) non-Abelian spinons and opposite chirality,

$$\begin{aligned}|\psi^{\rm S2CL}\rangle &= (\hat{\Psi}^{\rm S1CL}[a^\dagger, b^\dagger])^2|0\rangle \\ &= \hat{\Psi}_+^{\rm NACSL}[a^\dagger, b^\dagger]\hat{\Psi}_-^{\rm NACSL}[a^\dagger, b^\dagger]|0\rangle. \end{aligned} \quad (10)$$

The TD of the final state is ninefold (see Table I). It is hence an instance of a blocking mechanism for a nonchiral hierarchy state, which reduces the TD from $4 \times 4 = 16$ for the constituent S1CLs to 9. The construction as well as the TD suggests that (10) exhibits (Ising-type) non-Abelian spinon statistics for both chiralities.

The $S = 2$ chirality liquid state (10) is a promising candidate to capture a universality class of disordered $S = 2$ antiferromagnets, where the spin liquids may be stabilized through itinerant holes of appropriate kinetic energies, as the holon excitations in the hierarchical spin liquids presumably share the very high mobility of the holons of the individual constituent CSLs. The characteristic features of this universality class are, first, a $(S + 1)^2$-fold TD for the P and T invariant spin S hierarchy liquid on the torus, and second, that the spinons and holons obey non-Abelian SU(2) level $k = S$ statistics. The properties of the spinons would manifest themselves not only in the spin-liquid state, but in the response to all probes which measure energy scales beyond the ordering temperature, such as, e.g., Raman scattering.

State counting and effective field theory. There are important subtleties associated with the Schwinger boson projection scheme we employ here to obtain the hierarchy of spin liquids. The Hilbert space of an N-site spin $S = 1/2$ lattice contains 2^N states and is matched by the number of states in the configuration space of the spinon excitations that one can create in the CSL. For higher spin, however, the Schwinger boson projection maps a 2^{Nk}-dimensional product space to a $(k + 1)^N$-dimensional space in which the resulting spin liquid is defined. This poses the fundamental problem of how this reduction manifests itself for the spinons in the hierarchy liquids. For example, while the $S = 1$ chirality liquid (8) suggests a picture of free spinons of different chiralities, this picture can only be true at the lowest energies, since the state counting does not match. It will be interesting to investigate this issue further from a field theoretical perspective. Starting with an effective Chern-Simons theory for a single chiral spin liquid,[19] and the appropriate generalizations for the higher spin CSLs (9), we are led to conjecture that doubled Chern-Simons theories with the appropriate k may describe the low-energy physics for the non-Abelian higher spin chirality liquids. It is possible, however, that these theories have to be constrained, even though the number of consistent choices appears to be limited.[36]

Conclusion. We propose a hierarchy of spin liquids, which is constructed from chiral spin liquids as fundamental building blocks. We have shown that, in the P and T invariant liquids we obtain for spin $S = 2$ and higher, the topological degeneracies are reduced from what one would expect from the product of the constituent liquids. This provides an indication that the statistics of the spinons is non-Abelian. We present arguments indicating that the statistics for the P and T invariant spin S liquid is SU(2) level $k = S$. Finally, we conjecture that it might be possible to stabilize these liquids through doping with itinerant charge carriers.

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[1] X. G. Wen, Phys. Rev. B **40**, 7387 (1989).
[2] X. G. Wen and Q. Niu, Phys. Rev. B **41**, 9377 (1990).
[3] C. L. Kane and E. J. Mele, Phys. Rev. Lett. **95**, 226801 (2005).
[4] F. D. M. Haldane, Phys. Rev. Lett. **61**, 2015 (1988).
[5] F. Wilczek, *Fractional Statistics and Anyon Superconductivity* (World Scientific, Singapore, 1990).
[6] A. Stern, Nature (London) **464**, 187 (2010).
[7] B. I. Halperin, Phys. Rev. Lett. **52**, 1583 (1984); **52**, 2390(E) (1984).
[8] D. Arovas, J. R. Schrieffer, and F. Wilczek, Phys. Rev. Lett. **53**, 722 (1984).
[9] G. Moore and N. Read, Nucl. Phys. B **360**, 362 (1991).
[10] M. Greiter, X. G. Wen, and F. Wilczek, Nucl. Phys. B **374**, 567 (1992).
[11] G. E. Volovik, JETP Lett. **70**, 609 (1999).
[12] N. Read and D. Green, Phys. Rev. B **61**, 10267 (2000).
[13] P. A. Lee, Science **321**, 1306 (2008).
[14] L. Balents, Nature (London) **464**, 199 (2010).
[15] P. W. Anderson, Science **235**, 1196 (1987).
[16] S. Sachdev, Nat. Phys. **4**, 173 (2008).
[17] W. P. Su, J. R. Schrieffer, and A. J. Heeger, Phys. Rev. Lett. **42**, 1698 (1979).
[18] V. Kalmeyer and R. B. Laughlin, Phys. Rev. B **39**, 11879 (1989).
[19] X. G. Wen, F. Wilczek, and A. Zee, Phys. Rev. B **39**, 11413 (1989).
[20] M. Hermele, V. Gurarie, and A. M. Rey, Phys. Rev. Lett. **103**, 135301 (2009).
[21] R. Moessner, S. L. Sondhi, and E. Fradkin, Phys. Rev. B **65**, 024504 (2001).
[22] C. Xu and S. Sachdev, Phys. Rev. B **79**, 064405 (2009).
[23] A. Kitaev, Ann. Phys. **321**, 2 (2006).
[24] M. Greiter, J. Low Temp. Phys. **126**, 1029 (2002).
[25] Z. Zou, B. Doucot, and B. S. Shastry, Phys. Rev. B **39**, 11424 (1989).
[26] R. B. Laughlin and Z. Zou, Phys. Rev. B **41**, 664 (1990).
[27] A. Kitaev and J. Preskill, Phys. Rev. Lett. **96**, 110404 (2006).
[28] M. Levin and X.-G. Wen, Phys. Rev. Lett. **96**, 110405 (2006).
[29] F. D. M. Haldane and E. H. Rezayi, Phys. Rev. B **31**, 2529 (1985).
[30] D. Mumford, *Tata Lectures on Theta* (Birkhäuser, Basel, 1983), Vols. I and II.
[31] D. F. Schroeter, E. Kapit, R. Thomale, and M. Greiter, Phys. Rev. Lett. **99**, 097202 (2007); R. Thomale, E. Kapit, D. F. Schroeter, and M. Greiter, Phys. Rev. B **80**, 104406 (2009).
[32] M. Greiter and R. Thomale, Phys. Rev. Lett. **102**, 207203 (2009).
[33] M. Oshikawa and T. Senthil, Phys. Rev. Lett. **96**, 060601 (2006).
[34] N. Read and E. Rezayi, Phys. Rev. B **59**, 8084 (1999).
[35] X.-G. Wen and A. Zee, Phys. Rev. B **58**, 15717 (1998).
[36] T. H. Hansson, V. Oganesyan, and S. L. Sondhi, Ann. Phys. **313**, 497 (2004).