ELSEVIER

Physica E 1 (1997) 1–6

PHYSICA E

Quantum Hall quarks

Martin Greiter *

Department of Physics, Stanford University, Stanford, CA 94305, USA

Abstract

In order to obtain a local description of the short distance physics of fractionally quantized Hall states for realistic (e.g. Coulomb) interactions, I interprete the zeros of the ground-state wave function, as seen by an individual test electron from far away, as particles. I then present evidence in support of this view, and argue that the electron effectively decomposes into quark-like constituent particles of fractional charge.

Keywords: Quantum Hall states; Quarks

In this article, I will argue that electrons in fractionally quantized Hall fluids effectively decompose into smaller, quark-like particles, which then bind together to form electrons. This is not to say that electrons cease to be the fundamental degrees of freedom in these systems – a quantum mechanical description of all the electrons in the liquid is as complete as any description can be – but rather that the hierarchy of effective field theories is reversed. While we usually assume that constituent particles are more fundamental than composite particles – quarks are thought as more fundamental than hadrons in the standard model, or electrons as more fundamental than Cooper pairs in superconductors – fractionally quantized Hall liquids provide us with an example where the composite particles, the electrons, are fundamental while the smaller constituent particles, which I call *quantum Hall quarks*, are fictitious or effective degrees of freedom induced by the surrounding electron condensate.

* E-mail: greiter@quantum.stanford.edu.

I wish to address myself to readers without detailed knowledge of quantized Hall fluids, and will begin with a review of the long distance physics.

Most of our understanding of the fractionally quantized Hall effect is based on a highly original trial wave function for the ground state proposed by Laughlin [1]:

$$\Psi_m[z] = \prod_{i<j}^{N}(z_i - z_j)^m \prod_{i=1}^{N} e^{-(1/4)eB|z_i|^2}. \tag{1}$$

This wave function describes a circular droplet of an incompressible electron fluid in a strong perpendicular magnetic field B. The fact that all the electrons lie in the lowest Landau level constrains the wave function to an analytic function in the complex particle positions $z = x + iy$ times a Gaussian; the Jastrow factor $\prod(z_i - z_j)$ raised to an odd integer power m very effectively suppresses unwanted configurations in which electrons come close to each other.

The Landau level filling fraction is defined as $\nu \equiv \partial N/\partial N_\Phi$, where N is the number of electrons and N_Φ

1386-9477/97/$17.00
PII S1386-9477(97)00002-7

the number of Dirac flux quanta through the liquid. The latter is equal to the number of zeros of the wave function $\Psi[z]$ seen by an individual test electron with coordinate z_1 while all the other electron coordinates $z_2,\ldots,z_N$ are held at fixed positions. For the Laughlin state, Eq. (1) above, such a test electron will see m zeros at the positions of each other electron, and no additional zeros elsewhere. This implies $\nu = 1/m$.

The elementary excitations, quasiholes and quasielectrons, correspond to additional zeros which are not attached to electrons, or of deficits of zeros in given regions, respectively. Laughlin's explicit trial wave function for the quasihole is given by

$$\Psi_m^{\eta}[z] = \prod_{i=1}^{N}(z_i - \eta)\Psi_m[z]. \tag{2}$$

It is immediately obvious that m quasiholes at the same point η amount to a true hole in the liquid, which has charge $+e$; the convention here is $e>0$. The quasihole charge is therefore e/m. There is a similar trial wave function for the quasielectron, which involves derivatives in the z_i's.

The trial wave function, Eq. (1), is actually a rather good approximation to the exact ground state of two-dimensional electrons with Coulomb interactions in the lowest Landau level; at $\nu = \frac{1}{3}$, a numerical comparison for six electrons on a sphere yields [2, 3]

$$\langle \Psi_{m=3} | \Psi_{\text{exact}} \rangle = 0.9964. \tag{3}$$

The reason for this remarkable agreement, or more generally, for the success of Laughlin's theory, is that it captures the correct long distance physics. The essential physics contained in the trial wave function, Eq. (1) – in fact, the only physics except for the magnetic field – is that the electrons become *superfermions* for $m = 3,5,\ldots$. The notion of superfermions makes sense in two space dimensions only. It means that the phase picked up by the wave function when one electron encircles another is not 2π, as Fermi statistics requires it, but an odd multiple $2\pi m$, which is consistent with Fermi statistics as well. The fractional quantum numbers of the quasiparticles, for example, are a direct consequence of the superfermions.

Before closing this review, I would like to point out a technical detail [4] which will ease the exposition below. In the lowest Landau level, any two-body potential can be parameterized by a discrete set of pseudopotentials V_l, which denote the energy cost of having relative angular momentum l between two particles. The Laughlin $\frac{1}{3}$ state is the exact ground state of a model Hamiltonian where only the pseudopotential $V_1 > 0$ while all the other $V_l = 0$ for $l = 3,5,\ldots$. The reason for this is simply that the superfermions have – it follows directly from their definition – no amplitude to be in a state of relative angular momentum $l = 1$.

Now, imagine we adiabatically deform this set of pseudopotentials into the corresponding set for Coulomb interactions. Then the ground state will evolve from a Laughlin $\frac{1}{3}$ state into the exact Coulomb ground state at $\nu = \frac{1}{3}$. We know from the overlap, Eq. (3), that the state cannot change very much, and from the correctness of Laughlin's theory that the long distance physics cannot change at all – the changes must occur at short distances. The superfermions must evolve into *approximate superfermions*, that is, particles which look like superfermions from far away, yet are different from the exact superfermions contained in Laughlin's trial wave function.

To elucidate this notion, consider once more the zeros of the wave function as seen by an individual test electron z_1 while all the other electron coordinates $z_2,\ldots,z_N$ are fixed. The exact Coulomb ground state is of the general form

$$\Psi_{\text{Coul}}[z] = \prod_{i<j}^{N}(z_i - z_j)P(z_1,\ldots,z_N) \times \prod_{i=1}^{N} e^{-(1/4)eB|z_i|^2}. \tag{4}$$

The Jastrow factor must be present since $\Psi_{\text{Coul}}[z]$ is antisymmetric; $P(z_1,\ldots,z_N)$ is, in general, a complicated symmetric polynomial. A cartoon of the zeros of Ψ_{Coul} in a given region, as seen by a test electron from far away, is shown in Fig. 1.

There are three zeros associated with each electron: one of them (denoted by a cross) stems from the Jastrow factor in Eq. (4) and coincides with the electron coordinate z_i; the other two (denoted by dots) stem from the polynomial $P(z_1,\ldots,z_N)$ and are, in general, very complicated meromorphic functions of all the electron coordinates in a range which depends on the range of the interaction potential [5]. In the limit of the minimally short-ranged potential mentioned above, the positions of these two zeros depend only

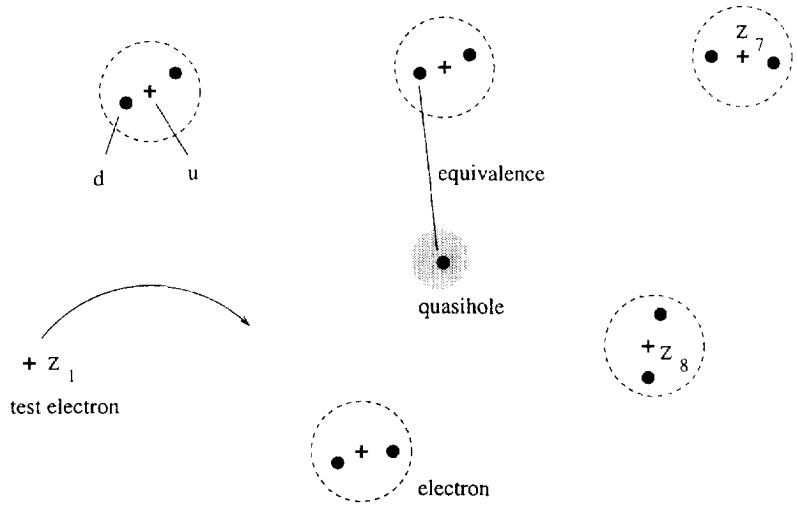

Fig. 1. Zeros of $\Psi_{\mathrm{Coul}}[z]$ as seen by an individual test electron z_1. The zeros denoted by crosses stem from a Jastrow factor and coincide with the electron positions $z_2,\ldots,z_N$, while those denoted by dots are in general very complicated functions of all the electron coordinates in the vicinity. Also shown is an isolated zero not associated with any electron, which corresponds to a quasihole excitation.

on the coordinate of the electron they are associated with – in fact, they coincide with this coordinate: $P(z_1,\ldots,z_N)$ becomes a Jastrow factor squared, and the general ground state, Eq. (4), the Laughlin $\frac{1}{3}$ state.

The reason the test electron must be far away is that the positions of the zeros associated with each electron depend on the position of the other electrons nearby. If we were to pick an electron nearby as a test electron, the zeros seen in this region would be those seen by another test electron from far away if the test electron nearby did not exist. The positions of the zeros would therefore depend on which of the electrons nearby we were to pick as a test electron. If we, however, choose an electron far away as the test electron, the positions of the zeros in the region nearby will not depend on our choice and an interpretation of the zeros as particle coordinates, as I will advocate below, is conceivable.

This brings me to the heart of the matter. In order to provide a *local* description of the short distance physics of fractionally quantized Hall fluids, I propose to view the zeros associated with the electrons as particles. The electron in a $\nu = \frac{1}{3}$ state effectively decomposes into three smaller constituent particles,

$$e^- \longrightarrow \mathrm{udd}, \tag{5}$$

where the u and d particles, or quantum Hall quarks, are the zeros due to the Jastrow factor and the polynomial $P(z_1,\ldots,z_N)$, respectively, as shown in Fig. 1. The d particles are equivalent to quasiholes, in the sense that a quasihole is nothing but an isolated d. The charge of the d must therefore be equal to the charge of the quasihole, which we know to be $+\frac{1}{3}$. Since the vacuum or ground state is neutral on a level on which the quasihole assumes this charge, the total charge of the udd composite must be zero, which implies that the charge of the u is $-\frac{2}{3}$.

The remainder of this paper is devoted to motivating and elucidating this idea. To begin with, I will use the hierarchy of quantized Hall fluids [4, 6, 7] to establish an interpretation of the quasiparticles in quantized Hall fluids as particles.

The quasiparticle excitations of quantized Hall liquids, quasielectrons and quasiholes, were originally conceived as vortices [1], and are adequately interpreted as such when a plateau in the Hall resistivity results from their localization by disorder. There are

situations, however, where an alternative interpretation as quantum mechanical particles is not only possible, but inevitable. The hierarchy of quantized Hall states provides us with an example: the quasiparticles *themselves* condense into a Laughlin–Jastrow-type fluid, and it is necessary to assign a wave function to them in order to describe this condensation. More precisely, we write an $[m,+p]$ state, that is a p daughter state of quasihole excitations of an m parent state, as [8, 9]

$$\Psi_{[m,-p]}[z] = \int D[\xi,\bar{\xi}]\Phi_p[\bar{\xi}] \times \prod_{k<l}^{N_1}(\xi_k - \xi_l)^{1/m}\prod_{k=1}^{N_1} e^{-1/(4m)eB|\xi_k|^2} \times \prod_{k=1}^{N_1}\prod_{i=1}^{N}(z_i - \xi_k)\Psi_m[z] \tag{6}$$

with the quasiparticle wave function

$$\Phi_p[\bar{\xi}] = \prod_{k<l}^{N_1}(\bar{\xi}_k - \bar{\xi}_l)^{p+1/m}\prod_{k=1}^{N_1} e^{-1/(4m)eB|\xi_k|^2} \tag{7}$$

and $N_1 = N/p$. The two factors in the second line of Eq. (6) serve to normalize the quasiparticle Hilbert space. The fact that we have to integrate over the quasiparticle coordinates to obtain a wave function for electrons is entirely consistent with their nature as quantum mechanical particles, as quantum mechanical degrees of freedom always have to be integrated out with a wave function as a measure whenever we wish to calculate a measurable quantity (e.g. a transition probability).

The explicit trial wave function, Eq. (6), and its cousin for the $[m,-p]$ state in which quasielectrons rather than quasiholes condense, are excellent approximations to the exact Coulomb ground states; at $\nu = \frac{2}{5}$, the overlap for 6 electrons on a sphere is [9]

$$\langle \Psi_{[3,-2]}|\Psi_{\text{exact}}\rangle = 0.9995, \tag{8}$$

a number which compares favorably even with the Laughlin $1/m$ states.

The particle nature of the quasiparticles leads us to the question of their origin, to the question of where new particles of fractional charge may come from. The answer is the obvious one, and this is precisely why it is so hard to swallow: The charges of the quasiparticles are parts of electron charges, and *the quasiparticles themselves are parts of electrons.* In order for quasiparticle excitations to exist, the vacuum or ground state must contain them already in a confined phase – the vacuum must be a phase in which pieces of electrons bind together to form electrons [10].

Particle physicists usually establish the existence of new particles by observing them as resonances in scattering experiments. This is not possible for quantum Hall quarks, because the kinetic energy of all the particles involved is quenched due to Landau level quantization, and the concept of time consequently does not exist, but even more profoundly so because we invoke quantum Hall quarks to describe the vacuum, which trivially excludes the possibility of scattering experiments.

Fortunately, there is a way around these problems. While we do not have a concept of real time, we can perform a Monte Carlo simulation and monitor scattering events as particle configurations evolve in Monte Carlo time. Let me briefly review the technique: a Monte Carlo simulation is a numerical method to approximate an integral over many variables with a probability ρ as a measure. Instead of integrating over the variables directly, we interpret them as dynamical variables, and let them evolve in Monte Carlo time. This concept of time is discrete; at each step we randomly pick one of the variables, and define a new configuration by randomly choosing a new value for this variable according to a certain distribution, which is usually taken as a Gaussian centered at the present value. Finally, we randomly decide whether to update the configuration or not according to probabilities proportional to the measure ρ for the new and for the present configuration, respectively. The desired integral is obtained by averaging the integrand (not including the measure) over a long span in Monte Carlo time; the approximation becomes exact as this span tends to infinity.

In our case, the Monte Carlo variables are the electron coordinates z_i, and the measure ρ is the probability $|\Psi_{\text{Coul}}(z_1,\ldots,z_N)|^2$. A snapshot of a typical Monte Carlo configuration including all the zeros or quantum Hall quarks, is shown in Fig. 2a. Only the electron coordinates, or u quantum Hall quarks, are truly dynamical variables in Monte Carlo time; the dynamics of the remaining zeros, or d quantum Hall quarks, is induced through the surrounding electron condensate. This, however, does not emerge from Fig. 2a, nor does it ever manifest itself as we follow the evolution of

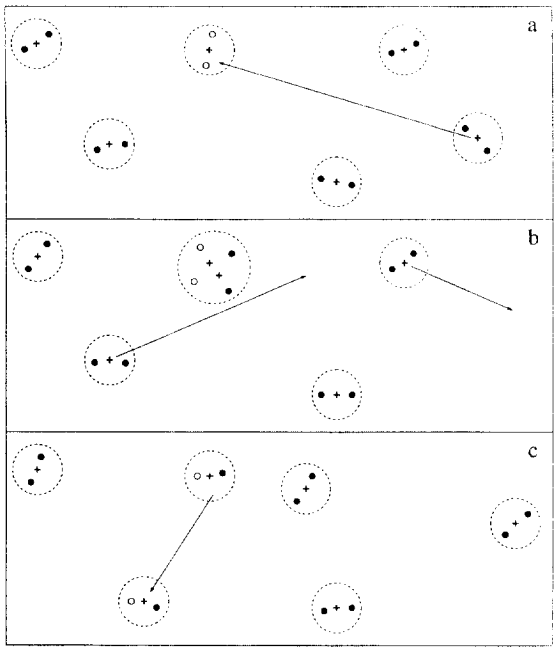

Fig. 2. Electron–electron scattering in Monte Carlo time: (a) two electron coordinates happen to come very close to each other; (b) the surrounding electron configuration evolves in Monte Carlo time, and with it the configuration of the zeros of the two electrons close to each other; (c) the two electrons separate again, having interchanged one of their constituent particles.

this configuration on a continuous time scale – that is, a time scale on which all the variables evolve simultaneously.

Let us now look at a particular scattering event, as shown in Fig. 2. In this event, two electrons scatter off each other, and interchange one of their constituent particles: two electron coordinates happen to come very close to each other, and remain unchanged for a number of Monte Carlo steps, while the configuration of the additional zeros associated with them evolves with the surrounding electron liquid; this configuration will, in general, have changed significantly by the time the two electrons separate again. Thus, there is a finite amplitude for zeros to get interchanged – the zeros are *indistinguishable* when interpreted as particles, and scatter into each other as identical particles do in quantum mechanics.

This Monte Carlo experiment nicely illustrates the underlying reason why it is possible for these fictitious or induced degrees of freedom to become particles: induced and fundamental degrees of freedom are *locally* equivalent, in the sense that no local experiment, and, in particular, no scattering experiment, is capable of resolving the difference. This is precisely the reason why it is perfectly reasonable to invoke quantum Hall quarks in order to provide a *local* description of fractionally quantized Hall fluids at short distances.

I have mentioned above that the d particle is equivalent to a quasihole excitation, in the sense that a quasihole is nothing but a d in isolation. To see this, we just need to perform another Monte Carlo experiment with an exact quasihole for Coulomb interactions at some location η, and we will find that the position of the zero associated with the quasihole does not exactly coincide with the position η, but rather depends on all the electron coordinates in the vicinity, as indicated in Fig. 1. Moreover, we will find that this zero has a finite amplitude to get interchanged with other zeros or d particle in the liquid as electrons scatter off the quasihole in Monte Carlo time. This illustrates the precise sense in which the exact quasihole for realistic interaction potentials differs from Laughlin's trial wave function, Eq. (2). The equivalence of confined and isolated zeros can of course also be deduced from the fact that a quasihole–quasielectron pair is created by removing a zero from the vacuum in a certain region and placing it into another region.

Most of what I have explained in this paper concerns the ground state or vacuum of fractionally quantized Hall fluids, while only excitations matter to experiments performed on quantum Hall systems. The real significance of the analysis presented here lies in the general message we can learn from it, and the potential relevance of this message to other systems, in particular, to the vacuum of our universe, the ground state which supports all the elementary particles known to us as excitations.

This general message is that some of the particles we see or detect as excitations above a certain vacuum might conceivably be pieces of larger particles invisible to us. The degrees of freedom we perceive as fundamental may, in fact, be fictitious or induced, and fractional quantum numbers – but, in particular, the fractional charges of quarks in quantum chromodynamics – may arise through a mechanism related to the one responsible for quantum Hall quarks. If we specifically imagine an observer who lives in a

quantized Hall fluid and consists of quasiparticles, this observer would never see electrons, but only fictitious particles of fractional charge, and would naturally be inclined to accept those as fundamental. Note, in particular, that scattering experiments, both the ones performed by this observer as well as the ones performed by us in particle accelerators, are incapable of resolving the ambiguity between induced and fundamental degrees of freedom.

A more elaborate account of this work will be given elsewhere [11].

I wish to thank L. Alvarez-Gaumé, E. Brézin, W. Krauth, R.B. Laughlin, L. Susskind and E. Verlinde for inspiring discussions, and to R.B. Laughlin again for his critical reading of the manuscript. This work was supported through a Fellowship in Elementary Particle Physics at CERN and through NSF grant No. DMR-95-21888.

References

[1] R.B. Laughlin, Phys. Rev. Lett. 50 (1983) 1395.
[2] F.D.M. Haldane, E.H. Rezayi, Phys. Rev. Lett. 54 (1985) 237.
[3] G. Fano, F. Ortolani, E. Colombo, Phys. Rev. B 34 (1986) 2670.
[4] F.D.M. Haldane, Phys. Rev. Lett. 51 (1983) 605.
[5] The corresponding observation for the effect of mixing with higher Landau levels has been made by B.I. Halperin, Helv. Phys. Acta 56 (1983) 75; my considerations here, however, only involve the lowest Landau level.
[6] R.B. Laughlin, Surface Science 142 (1984) 163.
[7] B.I. Halperin, Phys. Rev. Lett. 52 (1984) 1583.
[8] R.B. Laughlin, in: F. Wilczek (Ed.), Fractional Statistics and Anyon Superconductivity, World Scientific, Singapore, 1990.
[9] M. Greiter, Phys. Lett. B 336 (1994) 48.
[10] The possibility of an interpretation along these lines has been suggested by F. Wilczek, Liberating exotic slaves, Talk at the Celebration of the 60th Birthday of Y. Aharonov, IASSNS-HEP 94-58.
[11] M. Greiter, to be published.