

Paired Hall states

Martin Greiter ^{*}, X.G. Wen ^{**} and Frank Wilczek ^{*}

School of Natural Sciences, Institute for Advanced Study, Olden Lane, Princeton, NJ 08540, USA

Received 5 November 1991

Accepted for publication 18 December 1991

The principle that perturbation in quantum statistics should be accompanied by application of an appropriate magnetic field has been successful in giving a simple understanding of major qualitative features of the fractional quantized Hall states and related anyon superconducting states. In these applications, the starting point is one or more filled Landau levels. Here we consider the question of perturbation around free fermions. We argue that very near this point the statistical interactions are weak and their effects calculable; nevertheless they have the important qualitative consequence that a p-wave BCS pairing instability is triggered. The result is a new line of incompressible states in the (inverse) filling-fraction–statistics plane. This line extrapolates to a state obeying Fermi statistics at filling fraction $1/2$, which is a candidate to describe electron states. A variety of techniques is then employed to elucidate the properties of this state and the unusual quasiparticles it supports. We believe the state is in the same universality class as one Halperin proposed based on grouping electrons into pairs of tightly bound bosonic molecules, which form a correlated state of the Laughlin type. We report the results of extensive numerical work which establishes firmly the existence of an incompressible state with the properties we predict, including the very unusual quasiparticles, for simple model potentials. We also investigate the situation for realistic potentials, and conclude that a paired Hall state of the type investigated here is a good candidate to describe real 2d electron gases, especially for thick samples and higher Landau levels, quite possibly including the state at filling fraction $5/2$ that has already been observed.

1. Introduction and summary

Certainly one of the most startling and profound discoveries in recent condensed matter physics is the existence of incompressible ground states for the two-dimensional electron gas in a magnetic field, at favorable filling fractions (quantized Hall effect) [1,2]. The integer quantized Hall effect can be understood at least crudely on a one-particle picture, as resulting from the existence of Landau bands separated by energy gaps [3]. When the electron density is such as to

* Research supported in parts by DOE grant DE-FG02-90ER40542.

** Present address: Department of Physics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA.

precisely fill an integer number of levels, any rearrangement of the density requires the existence of a hole somewhere and an excitation to the next band somewhere else. This clearly requires an irreducible, finite injection of energy. (A proper understanding of why the state can accommodate finite changes in density – in experimental terms, of why there is a *finite* rather than an infinitesimal plateau in the quantized resistivity – requires an additional argument. It must be shown that small deviations from the ideal density can be accommodated on localized states pinned to impurities, which do not contribute to the current flow.) The fractional quantized Hall effect [4–7], by contrast, *contradicts* the one-particle picture. The one-particle picture would lead one to expect a vast sea of degenerate or nearly degenerate states, with no gap, when the valence Landau band is partially filled. Thus non-trivial correlations among the electrons must be invoked to understand the existence of incompressible states at fractional fillings, even crudely.

The existence and form of these correlations has been convincingly established, at least for the primary states at $\nu = 1/(2m + 1)$, by a variety of arguments. Perhaps conceptually the simplest and clearest (though by no means the first or only) way to understand the existence of incompressible states at these fractions is the adiabatic approach suggested by two of us recently [8,9]. According to this approach, which we shall elaborate below, one can continuously interpolate between $\nu = 1$ and $\nu = 1/(2m + 1)$ through a series of incompressible states. In this process, one changes the hamiltonian by replacing the uniform magnetic field by an equal amount of flux localized in point-like tubes attached to the particles, and also adds suitable potentials (which are necessary to keep the gap from closing.) Thus one relates incompressible states with different quantum statistics in different background magnetic fields. At $\nu = 1/(2m + 1)$ one has incompressible fermion states, suitable for describing ordinary electrons, while at intermediate values one has incompressible anyon states. The existence of incompressible states at $\nu = 1/(2m + 1)$ is then a consequence of the adiabatic theorem; these states descend, in the simple and precise sense indicated, from the full Landau level.

One might try to consider, instead of a single filled Landau level, other starting points for this adiabatic flux-trading procedure. Perhaps the most basic and interesting starting point of all is free fermions in zero magnetic field [10]. Unfortunately it is not at all obvious that an adiabatic procedure with this starting point makes sense, since the filled fermi sea is not an isolated state – there are many low-energy excitations. However, it is a very familiar fact that free fermions are poised on the brink of an instability – the BCS pairing instability. Arbitrarily weak interactions can trigger this instability, which does open a gap. It makes sense to attempt an adiabatic evolution, similar to the one used to obtain the $\nu = 1/(2m + 1)$ states from the full Landau level, starting from a paired state in zero magnetic field. Doing this, we find candidate fermion states at $\nu = 1/2m$. These states, and in particular the simplest one at $\nu = 1/2$, are the main subject of this paper.

Some time ago Halperin [5] suggested the possibility in principle of incompressible fermion states at $\nu = 1/2$, based on considerations different from but not entirely unrelated to these. He invites us to imagine, for purpose of discussion, that there is some powerful attractive but saturable force, that organizes charged fermions into tightly bound pairs, with repulsive residual interactions among the bound pairs. Then we have the conditions to form an incompressible *boson* state of Laughlin type among the pairs. For bosons, the primary filling fractions are of the form $\nu = 1/2p$. Since these bosons have twice the charge and half the density of the original fermions, it is not difficult to see that $p = 4$, an allowed value for the effective bosons, corresponds to $\nu = 1/2$ for the original fermions.

Of course, this approach begs the question of where such a bizarre force might originate, and whether it is reasonable to expect electrons, which are basically repulsive, to form effective pairs. As we shall argue in detail below, there is every reason to believe that the states we describe are in the same universality class as the ones Halperin suggested, thus in a sense validating his discussion a posteriori. The relationship between the theories is quite similar to the relation between the Bose condensation picture of superconductivity and BCS theory. For many qualitative purposes it is simpler, and acceptable, to use the tight-binding picture. However this picture is far from justified quantitatively. It also misses out on the neutral pair-breaking excitations, which we believe may be quite important for the $\nu = 1/2$ state in reality.

Now let us briefly discuss the literature and status of paired Hall states. As we have mentioned, the possibility of paired Hall states was raised by Halperin, whose strong-pairing picture also provides a simple way to envision many of their qualitative properties. Although they did not interpret them this way, numerical experiments of Canright and Girvin [11] demonstrated the existence of incompressible states at filling fraction $\nu = 1$ for repulsive spinless bosons, which we strongly suspect are paired Hall states. Moore and Read [12] emphasized the usefulness of pfaffians in constructing trial wave functions with pairing correlations in the context of the Hall effect. Although (as discussed below) we have reservations concerning their proposal of non-abelian statistics for the associated states, we arrive at the same trial wave functions for ground state and charged quasiparticle excitations. Moore and Read were largely motivated by far-reaching analogies between conformal field theory correlators and the ground state wave functions of quantized Hall states. We were initially motivated to consider pfaffians by the fact that the BCS pairing wave functions, in real space, take the form of a pfaffian. We will demonstrate below that it is generated by adiabatic evolution in quantum statistics from an extremely simple (exactly soluble, but singular) BCS superconductor, and also by exact solution of a simple local Hamiltonian with short-range three-body repulsions. The fact that such different points of view all lead to the same class of wave functions, certainly adds to their interest and credibility. There is a well-established Hall state at $\nu = 5/2$ [13], and hints of anomalous behavior at

other even denominators [14]. These do not fit in to the standard hierarchical construction [6] for spin-polarized electrons. Trial wave functions incorporating non-trivial admixtures of different spins for the electrons have been proposed for these states [5,15], but the most recent and detailed numerical studies have tended to favor fully polarized states [16,17]. It has been shown that the $5/2$ state can be destroyed by a strong perpendicular component in the magnetic field [18], which at first sight certainly suggests the relevance of spin correlations. However introduction of the perpendicular field introduces other effects beside a Zeeman splitting, and in our opinion it would be premature to regard the issue as closed. In light of the numerical results mentioned, and additional ones presented below, we regard the polarized paired Hall state as excellent candidates to describe real states of matter, including the $5/2$ state.

The contents of the remainder of this paper are as follows. In sect. 2, we very briefly review the adiabatic procedure, and argue that it suggests a connection between the BCS pairing instability for free fermions and an incompressible Hall state (for spinless or spin-polarized fermions) at filling fraction $\nu = 1/2$. In sect. 3, we analyze the pairing instability induced by residual statistical interactions in detail. We argue that it leads unambiguously to pairing in the p-wave for weak coupling (that is, small deviations from fermi statistics at zero external field.) Unfortunately, the most interesting case $\nu = 1/2$ is far from this limit, and other approaches must be used. Fortunately, the pairing analysis itself suggests a particular trial wave function, involving the mathematical object known as a pfaffian, which has the required qualitative features and (we shall see) is exact for one class of interactions. In sect. 4 we discuss the expected “quantum numbers” of the ground state. These are whole numbers having to do with the degeneracy, and the relationship between charge and flux, on closed surfaces. We first give a qualitative discussion, based on the strong pairing picture. We then give an alternative analysis, based on the suggested trial wave function. This requires generalizing the droplet wave function to the sphere and torus, which proves quite instructive. In sect. 5, we discuss the charged quasiparticle excitations above the paired Hall state. (Functions for the halberons.) We argue that the fundamental charged quasiparticles are charge $e/4$ anyons with statistical parameter $\theta = \pi/8$, and provide trial wave functions for these quasiparticles (here christened halberons). Their form on a torus is particularly interesting, and has non-trivial relations both to the ground state degeneracy and to the existence of pair-breaking modes. In sect. 6, we discuss the neutral fermion (pair-breaking) excitations. It is argued that the existence of such excitations allows one to evade the conclusion of Tao and Wu [19], that only odd denominators are allowed for the fractional quantized Hall effect with spin-polarized electrons. In sect. 7, we discuss some numerical experiments, which give evidence that a state of the kind we are discussing is the ground state for suitable interaction potentials. This includes evidence based on the relation between flux and particle number on a sphere

(which includes a characteristic offset), and numerical evidence for the suggested properties of the quasiparticles. In sect. 8, we consider the question whether a paired Hall state is the ground state for realistic or accessible potentials. It does not appear to be the ground state for the simple Coulomb potential (projected to the lowest Landau level), but we find that after inclusion of higher Landau level correlations and finite thickness effects it becomes a compelling candidate to describe real states of matter, quite possibly including some that have already been observed.

To avoid repetition let us state once and for all that in this paper unless stated otherwise we will implicitly ignore the electron spin, so that we deal with effectively scalar fermions. This is presumably appropriate at least for very strong magnetic fields, in circumstances where the Zeeman splitting lifts the “wrong” spin above the energy scale of interest.

2. Search for analyticity in statistical perturbations

The possibility of anyon statistics allows one to consider perturbing or interpolating in particle statistics in two space dimensions [20,21].

Straightforward perturbation theory is likely to be problematic, however, for the following reason. Small changes in the statistics correspond to putting a small quantity of fictitious magnetic flux on each particle. Typical calculations one might want to contemplate would be to project out the ground state by the standard device (Feynman) of starting from an arbitrary configuration and letting the system evolve for a large imaginary time, or to calculate the partition function by summing over all periodic paths in imaginary time, with period inversely proportional to the temperature. In either case, insofar as the important paths wander over many interparticle spacings, they acquire their phase from winding around many tubes each individually responsible for only a small phase. One would therefore obtain almost the same result by replacing the flux localized on the particles by its uniform average. Thus if we wish to treat the change in statistics as a small residual interaction, then at least under the condition that the paths wander – that is, in the regime of strong quantum phenomena, or low temperature – the proper starting point must be to analyze the problem in the appropriate background magnetic field. Only then will the residual interactions be in any sense small or local. At high temperatures or low density this consideration is not so critical, and one does find more or less smooth behavior in the quantum statistics parameter, although even in this case a cusp in the second virial coefficient arises in the first-order perturbation around bosons [22].

This discussion is closely related to, and is reinforced by, a heuristic principle recently emphasized by two of us [8,9]. It was proposed that incompressible states

along the lines

$$\Delta \frac{\theta}{\pi} = \Delta \frac{1}{\nu} \quad (2.1)$$

in the statistics–magnetic field (or inverse filling fraction) plane are likely to be adiabatically related, by the procedure of slowly trading flux localized on the particles for an equal amount of uniform flux. This procedure generalizes the successful RPA treatment [23,24] of anyon superconductors at $\theta/\pi = (1 - 1/n)$ in zero magnetic field by perturbation around fermions ($\theta/\pi = 1$) filling n Landau levels ($\nu = n$). It also relates, for $\Delta(\theta/\pi) = \text{even integer}$, integer to fractional quantized Hall states of fermions – the connection being made through a continuous succession of anyon states. This observation forms the basis of a systematic perturbative approach to the latter states, which we plan to discuss at length elsewhere. However at this moment a *negative* consequence of the heuristic principle is more relevant. That is, that this principle makes it clear that perturbing in θ/π *without* changing the magnetic field is a highly non-analytic procedure. For according to the heuristic principle this perturbation is smoothly related to a perturbation in $1/\nu$ without changing the quantum statistics – and we know that this sort of perturbation is extremely non-analytic; in fact the existence of cusps in energy as a function of filling fraction is the essence of the fractional quantized Hall effect.

In this paper we wish to focus on another case where the adiabatic heuristic, or the idea of smooth perturbation in quantum statistics suggests that particularly interesting behavior will occur. We ask: what states are continuously related to free fermions in *zero* magnetic field? This is the reference problem for anyons with statistical parameter $\theta = \pi(1 + \epsilon)$ in the presence of a uniform magnetic field of magnitude $B = 2\pi\epsilon\rho/e$. Because it takes its point of departure from free fermions, the line $1/\nu = (\theta/\pi) - 1$ was originally called the line of anyon metal. It seems now more appropriate to call it the line of paired Hall states.

One reason that this line is especially interesting is that the reference problem – free fermions in zero field – is poised on the brink of an instability. We have in mind the pairing instability of BCS theory [25], which can be triggered by arbitrarily weak attractions. For this reason, the residual interactions may have a large effect even in the small- ϵ regime where they are small and their effects are reliably calculable. In fact we shall soon argue that the residual interactions do trigger the pairing instability. They give us a p-wave superconductor – or, to be more precise, pairing correlations in the p-wave. Moreover, the strength of the pairing *increases* with ϵ . Thus although the perturbative calculation becomes unreliable, it still strongly suggests that this type of ordering occurs for large ϵ as well.

Travelling along this line in statistics – magnetic field space, one reaches an interesting point at $\epsilon = 1$. At this point one has reached a (presumably incompress-

ible) state of *bosons* exactly filling one Landau level, $\nu = 1$. We find it quite unusual and amusing that bosons can recognize the significance of exactly filling one Landau level. There is some numerical evidence [11] that bosons with hard core repulsion do in fact organize themselves into an incompressible state at $\nu = 1$. Although a direct physical realization does not seem likely, this boson state is likely to be a good testing ground for the ideas described here. We are claiming its behavior is qualitatively similar to the physically realizable fermion state at $\nu = 1/2$, but it ought to be more tractable numerically and analytically.

Travelling still further along our line in statistics-magnetic field space, one reaches an especially interesting point at $\epsilon = 2$. At this point the statistical parameter has come full circle to arrive back at fermions. Thus fermions in a magnetic field at filling fraction $\nu = 1/2$ are related to fermions in zero magnetic field with residual local interactions. This is interesting, because a fermion state at least meets the most basic requirement for a candidate to describe interacting electrons. Since this particular fermion state has been evolved continuously out of an adiabatic procedure from an incompressible (gapped) system, and is therefore very plausibly the ground state for suitable interaction potentials – i.e. repulsive potentials such that the gap does not close at intermediate stages.

Unfortunately, $\epsilon = 2$ is beyond the bounds of perturbation theory. For $\epsilon = 2$ the residual interactions, though local, are strong. Nevertheless, one might hope to learn something about the possible behavior of fermions at $\nu = 1/2$ by extrapolating the calculable behavior for small ϵ . To this, we now turn.

3. The pairing instability and the ground state

3.1. THE RESIDUAL INTERACTION AND THE INSTABILITY

In this section we shall derive the residual interaction as we perturb from free fermions into statistics – magnetic field space, and argue that this interaction does indeed trigger a pairing instability.

For convenience, let us use second quantized notation. Then the hamiltonian for an anyon gas in a uniform, flux-compensating magnetic field is given by

$$H = \frac{1}{2m} \int d^2r \Psi^\dagger(r) [p + a(r)]^2 \Psi(r), \quad (3.1)$$

with

$$a(r) = \epsilon \int d^2r' \frac{\hat{z} \times (r - r')}{|r - r'|^2} \{ \Psi^\dagger(r') \Psi(r') - \bar{\rho} \}. \quad (3.2)$$

Here Ψ is a spinless fermion field (in accordance with the assumption that the $\nu = \frac{1}{2}$ state is spin polarized), and $\bar{\rho}$ is the mean particle density. Our reference problem for perturbation theory is, of course, the idealized case of free fermions. The residual interaction is

$$H_{\text{int}} = H - H_0 = \frac{1}{2m} \int d^2r \Psi^\dagger(r) \{2pa - a^2\} \Psi(r), \quad (3.3)$$

or, if we neglect terms of second order in ϵ

$$H_{\text{int}} = \frac{\epsilon}{m} \int \int d^2r d^2r' \Psi^\dagger(r) \frac{p \times (r - r')}{|r - r'|^2} \Psi(r) \{ \Psi^\dagger(r') \Psi(r') - \bar{\rho} \} \quad (3.4)$$

The velocity-dependence of the two body interaction,

$$H_{\text{int}} \sim \frac{v \times r}{|r|^2}$$

already hints that the ground state will not be an s-wave, but a higher angular momentum eigenstate capable of taking advantage of this potentially attractive interaction. Following the standard BCS analysis, we rewrite the hamiltonian in Fourier space and keep only those terms decisive for superconductivity

$$H_{\text{int}} = \sum_{k, k'} c_{k'}^\dagger c_{-k}^\dagger V_{kk'} c_{-k} c_k, \quad (3.5)$$

where the interaction is given by

$$V_{kk'} = 2\pi i \frac{\epsilon}{m} \frac{k \times k'}{|k - k'|^2}. \quad (3.6)$$

We now minimize the expectation value of this hamiltonian with respect to a BCS pairing wave function. By familiar steps one is led to the consistency equation

$$\Delta_k = -\frac{1}{2} \sum_{k'} \frac{\Delta_{k'}}{E_{k'}} V_{kk'}, \quad (3.7)$$

for the superconducting gap parameter. E_k is the quasi-particle excitation energy, defined by

$$E_k \equiv \sqrt{|\Delta_k|^2 + \xi_k^2}, \quad (3.8)$$

with

$$\xi_k \equiv \epsilon_k - \epsilon_f = \frac{1}{2m} (k^2 - k_f^2),$$

where ϵ_f denotes the Fermi-energy. Now let us discuss the solution of the gap equation (3.7).

To begin with, we substitute the expression (3.6) for the interaction potential $V_{kk'}$ into the gap equation and replace the sum over k' by an integral:

$$\Delta_k = -\pi i \frac{\epsilon}{m} \frac{1}{(2\pi)^2} \int d^2k' \frac{\Delta_{k'}}{E_{k'}} V_{kk'} \frac{k \times k'}{|k - k'|^2}. \quad (3.9)$$

The imaginary prefactor indicates that Δ_k cannot be real, as it would be for an s-wave superconductor. We are led to consider the Ansatz for l -wave pairing

$$\Delta_k = |\Delta_k| e^{i\varphi_k l} \quad \text{with } l = 1, 2, 3, \dots, \quad (3.10)$$

where φ_k denotes the direction of the two-vector k . Note that only odd values for l are consistent with Fermi statistics (and the assumption that the state is spin-polarized). Substituting this Ansatz into the gap equation (3.9), we find

$$|\Delta_k| = \frac{\epsilon}{4\pi m} \int_0^\infty k' dk' \frac{|\Delta_{k'}|}{E_{k'}} I_l \left(\frac{1}{2} \left(\frac{k'}{k} + \frac{k}{k'} \right) \right), \quad (3.11)$$

where

$$I_l(\lambda) \equiv \int_0^{2\pi} d\varphi \frac{\sin l\varphi \sin \varphi}{\lambda - \cos \varphi}. \quad (3.12)$$

This integral can be evaluated, and yields

$$I_l(\lambda) = \begin{cases} 0 & \text{for } l \text{ even} \\ 2\pi(\lambda - \sqrt{\lambda^2 - 1})^l & \text{for } l \text{ odd,} \end{cases} \quad (3.13)$$

provided $\lambda \geq 1$. If we substitute (3.13) for l odd into eq. (3.11), we obtain

$$|\Delta_k| = \frac{\epsilon}{4\pi m} \left\{ \int_0^k k' dk' \left(\frac{k'}{k} \right)^l \frac{|\Delta_{k'}|}{E_{k'}} + \int_k^\infty k' dk' \left(\frac{k}{k'} \right)^l \frac{|\Delta_{k'}|}{E_{k'}} \right\}. \quad (3.14)$$

Eq. (3.14) can be solved almost exactly for small ϵ . One very significant result can be discerned immediately. The integral on the right-hand side of eq. (3.14) acquires its largest value for $l = 1$, and consequently the pairing for the ground state must be p-wave. Therefore we take $l = 1$ in the remainder of the discussion.

Further insight can be gained if we reformulate the integral equation (3.14) as a differential equation:

$$\left\{ k^2 \frac{\partial^2}{\partial k^2} + k \frac{\partial}{\partial k} + \left(\epsilon \frac{k^2/2m}{E_k} - 1 \right) \right\} |\Delta_k| = 0, \quad (3.15)$$

where E_k is still a function of $|\Delta_k|$, as given in (3.8). We expect $|\Delta_k| \ll \epsilon_f$ for small ϵ – an assumption which will be justified a posteriori. The term containing E_k is of order ϵ and can be neglected, provided that k is not too close to the Fermi-momentum k_f . The remaining homogeneous differential equation has two independent solutions:

$$|\Delta_k| \propto k \quad \text{and} \quad |\Delta_k| \propto \frac{1}{k}. \quad (3.16)$$

From the integral equation (3.14), it follows clearly that the gap parameter must vanish for both $k = 0$ and $k = \infty$. Thus we can anticipate the shape of the solution everywhere except at k_f . The exact behavior at this point, however, is irrelevant for our purposes, since none of the important parameters depends on it. So the essential k -dependence is given by

$$|\Delta_k| = |\Delta_{k_f}| \times \begin{cases} |k/k_f| & \text{for } k < k_f \\ |k_f/k| & \text{for } k > k_f. \end{cases} \quad (3.17)$$

In order to find the remaining parameter $|\Delta_{k_f}|$, we substitute the solution (3.17) back into the integral equation (3.14), set $k = k_f$ and do the integrals over k' on the right-hand side. Assuming $\epsilon \ll 1$, of course, and also $|\Delta_{k_f}| \ll \epsilon_f$, we obtain to an excellent approximation

$$|\Delta_{k_f}| = \frac{2}{\sqrt{e}} \epsilon_f e^{-2/\epsilon}. \quad (3.18)$$

Thus the latter assumption is manifestly valid if we presuppose the former. The solution (3.18) becomes exact in the limit $\epsilon \rightarrow 0$.

The critical temperature, defined as the temperature at which the gap closes, also can be evaluated in close analogy to the classic BCS analysis. Within the regime of the assumptions made above, one obtains

$$k_B T_c = \frac{1.13}{\sqrt{e}} \epsilon_f e^{-2/\epsilon}. \quad (3.19)$$

Consequently, the gap parameter at $k = k_f$ and the critical temperature are related by

$$|\Delta_{k_f}| = 1.76 k_B T_c,$$

as familiar from conventional BCS superconductivity.

3.2. WAVE FUNCTIONS

The analysis in the preceding section shows, that the residual statistical interactions induce pairing correlations of the BCS type for weak coupling – that is, in our context, for small perturbations from Fermi statistics. However the greatest physical interest focuses on large perturbations, and ultimately on a perturbation so large as to bring us full circle, back to fermions. Straightforward perturbation theory is of course not adequate to this extrapolation. However the pairing does suggest a definite trial wave function for our problem, as follows.

Since we are concerned with problems in a magnetic field, we may wish to restrict ourselves to the lowest Landau level. The restriction to the lowest Landau level is by far most easily implemented in real space, where it simply tells us that the wave functions are analytic in the complex particle positions. Thus we should look for analytic wave functions that incorporate BCS pairing in real space. BCS theory is usually formulated in momentum space, for good reasons (the Fermi surface, which is central to the superconductivity of metals, is most easily located in momentum space). However it is a classic, though perhaps not widely known, fact that the BCS wave function in real space has a rather simple and special form. It can be written

$$\Psi_{\text{BCS}}(z_1, z_2, \dots, z_{2n}) = \mathcal{A} \prod_{i \text{ even}}^N \phi(z_{i-1} - z_i) \quad (3.20)$$

where we have already taken into account that Ψ_{BCS} is meant to be analytic in the complex particle positions. The operator \mathcal{A} indicates that one should antisymmetrize, over all $(N-1)!!$ different possible ways of dividing N particles up in to pairs (N is assumed even), the product of $(N/2)$ factors of ϕ . This form can be obtained from the more conventional momentum space representation by projecting onto a definite number of particles, as demonstrated by Dyson [26] a long time ago. It is closely related to the mathematical object known as a pfaffian.

The function ϕ – which must be odd, to be consistent with fermi statistics – is to be interpreted as the wave function for the relative coordinate within each pair. For a sensible physical interpretation, we must demand that the pair wave function represent attraction between the members of the pair – that is, it should be a monotone decreasing function of their separation. The simplest and least singular wave function of this type is of course

$$\phi(z) = 1/z. \quad (3.21)$$

This simplest choice corresponds to pairing correlations in the p-wave sector, and is thus consistent with the perturbative results obtained in sect. It has the apparent difficulty that it is singular at $z=0$, which implies a non-normalizable wave function diverging for small separations. However in the context of the quantized

Hall effect it is common to consider attaching fictitious charge and flux to the electrons; or, alternatively, what amounts to the same thing, to consider wave functions including positive powers of the differences $z_i - z_j$ for every set of two particles (not just the paired ones). The appearance of such powers removes the objection to the singularity of the natural analytic pair wave function, since the complete wave function, in which it is one factor, is not singular.

The rough idea of attaching flux to particles, and its connection to specific wave functions, gains precision in the context of some simple soluble models, to which we now turn.

3.3. AN EXACT SOLUTION AND ITS EXTRAPOLATION

In this section we shall elaborate on an exact model, which allows us to explicitly carry through the process of adiabatic extrapolation in quantum statistics [9,27]. This will provide us not only with a most plausible (and essentially unique, as we shall argue in sect. 4) form for the wave function at half filling, but also with strong evidence for one of the most decisive properties of the Hall liquid: the existence of a pairing instability in the absence of an attractive interaction.

In fact, the initial ground state of our model is given by the unnormalizable BCS wave function motivated above. It is an exact zero energy eigenstate of the N -particle hamiltonian

$$H_{\text{BCS}} = \frac{1}{2m} \sum_j (-i\nabla_j)^2 - \frac{\pi}{m} \sum_{i \neq j} \delta^2(\mathbf{r}_i - \mathbf{r}_j). \quad (3.22)$$

This hamiltonian describes a (spinless) BCS superconductor in position space, for the attractive δ -function potential here is equivalent to a constant potential in momentum space. It is most convenient to write the ground state using a pfaffian

$$\Psi_{\text{BCS}} = \text{Pf} \left(\frac{1}{z_i - z_j} \right), \quad (3.23)$$

where

$$\text{Pf} \left(\frac{1}{z_i - z_j} \right) = \mathcal{A} \prod_{i \text{ even}}^N \frac{1}{z_{i-1} - z_i}. \quad (3.24)$$

The exactness of this model may easily be verified directly, with the identity

$$\nabla^2 \frac{1}{z} = -2\pi \frac{\delta^2(z)}{z}.$$

which is to be interpreted as a prescription for integrating smooth functions which

vanish at the origin, slightly generalizing the usual definition of distributions. (The delta function is of course meant to be taken over the real and imaginary part separately.)

Since Ψ_{BCS} is not normalizable, it is without evident physical meaning by itself. Formally, it attempts to describe a BCS superconductor with a short range pairing potential so strong that the potential energy gained is large enough to compensate entirely for the total kinetic energy of the system.

The virtues of the singular solution become most apparent only as we initiate the adiabatic process described above. The model will remain exactly soluble, provided we also vary the strength of the delta-function interaction, and replace the original hamiltonian by

$$H_\epsilon = \frac{1}{2m} \sum_j (-i\nabla_j - eA_j)^2 - \frac{\pi(1-\epsilon)}{m} \sum_{i \neq j} \delta^2(\mathbf{r}_i - \mathbf{r}_j), \quad (3.25)$$

where

$$A_j = \frac{\epsilon}{e} \sum_{i \neq j} \frac{(\mathbf{r}_i - \mathbf{r}_j) \times \hat{z}}{|\mathbf{r}_i - \mathbf{r}_j|^2} + \frac{1}{2} B_\epsilon (\mathbf{r}_j \times \hat{z}). \quad (3.26)$$

A_j implements both fractional statistics with parameter $\theta = \pi(1 + \epsilon)$ and a flux-compensating, uniform magnetic background field $B_\epsilon \equiv 2\pi\epsilon\rho/e$ in the negative \hat{z} -direction. The exact ground state evolves into

$$\Psi_\epsilon = \text{Pf} \left(\frac{1}{z_i - z_j} \right) \prod_{j < k} |z_j - z_k|^\epsilon \prod_j \exp \left(-\frac{1}{4} e B_\epsilon |z_j|^2 \right). \quad (3.27)$$

The energy of this state is numerically equal to the kinetic energy of N particles in the lowest Landau-level (i.e. $E_\epsilon = \frac{1}{2} \omega_c N$, with $\omega_c = (e/m)B_\epsilon$).

The final point $\epsilon = 2$ is particularly interesting. The statistics has then evolved back to fermions, and the magnitude of the external magnetic field is such that the filling fraction is one-half. The flux-tubes attached – two Dirac flux quanta on each particle – are no longer of physical significance. In fact, they may be removed via a singular gauge transformation:

$$\begin{aligned} A_j &\rightarrow A_j + \nabla_j \Lambda_j \quad \text{for all } j, \\ \Psi_\epsilon &\rightarrow \Psi_\epsilon \prod_j \exp(-ie\Lambda_j), \end{aligned} \quad (3.28)$$

with

$$\Lambda_j = \frac{1}{e} \sum_{i \neq j} \arg(z_i - z_j). \quad (3.29)$$

We say that this transformation is singular, because A_j becomes ill-defined if any one of the other particles takes the same position as the j th particle. However, a centrifugal barrier excludes this possibility, and we may safely remove these points from the N -particle configuration space.

Thus we obtain the final hamiltonian

$$H_{\text{phs}} = \frac{1}{2m} \sum_j (-i\nabla_j - eA_j)^2 + \frac{\pi}{m} \sum_{i \neq j} \delta^2(\mathbf{r}_i - \mathbf{r}_j), \quad (3.30)$$

where

$$A_j = \frac{1}{2} B(\mathbf{r}_j \times \hat{z}). \quad (3.31)$$

H_{phs} differs from our starting point H_0 only in two respects: through the presence of a uniform magnetic background field, and through the sign of the delta function interaction. Removal of the flux tubes affects the ground state only by a phase, and yields

$$\Psi_{\text{phs}} = \text{Pf} \left(\frac{1}{z_i - z_j} \right) \prod_{j < k} (z_j - z_k)^2 \prod_j \exp \left(-\frac{eB}{4} |z_j|^2 \right). \quad (3.32)$$

Note that Ψ_{phs} is an entire function of the complex particle positions multiplied by an appropriate exponential factor, and thus a product of single particle states in the lowest Landau level. It represents a pairing state at filling fraction one-half: a Laughlin state modulated by a strong attractive pairing correlation – which is to say, the very strong anti-correlation implicit in the Laughlin wave function is partially ameliorated. The filling factor in the thermodynamic (large- N) limit is insensitive to the pairing, as can be seen from a simple angular momentum argument.

Pairing is conventionally associated with an attractive interaction. Our exact solution, however, makes it very clear that this association is not inevitable. As we travel through statistics – magnetic field space, the coefficient of the delta-function potential in (3.25) changes continuously. It is attractive for the unnormalizable superconductor, vanishes as we pass through boson statistics, and remains repulsive as we reach the Hall state (strictly speaking, however, it is significant only when it is attractive). This implies that the pairing in the Hall-state does *not* require an attractive interaction; rather it arises indirectly as a necessary accessory of Jastrow–Laughlin correlations at an even-denominator filling fraction.

3.4. PHYSICAL UNIQUENESS OF THE DROPLET WAVE FUNCTION

Our droplet wave function (3.32) is an example of a class of wave functions

$$\Psi_{1/2} = \psi_1 \psi_2, \quad (3.33)$$

where ψ_1 is of the BCS–Dyson form

$$\psi_1 = \mathcal{A} \prod_{\text{pairs}(ij)} \phi(z_i - z_j). \quad (3.34)$$

and ψ_2 is of the Laughlin–Jastrow type form

$$\psi_2 = \prod_{i < j} (z_i - z_j)^2 \prod_i \exp\left(-\frac{eB}{4} |z_i|^2\right). \quad (3.35)$$

These wave functions are of the simplest form, that contains both pairing correlations and the characteristic short-range repulsions of the Laughlin function.

Now we shall argue that the choice made above,

$$\phi(z) = 1/z, \quad (3.36)$$

is not only the simplest and most appealing, but essentially unique. Suppose instead that we chose $\phi(z)$ to be an odd polynomial, say of degree k . Then the pfaffian would be a totally antisymmetric polynomial in N variables, of degree $kN/2$. However a totally antisymmetric polynomial in N variables must contain a factor $z_i - z_j$ for every pair i, j . (That is, it must contain the Vandermonde determinant as a factor.) If it is not to vanish, such a polynomial must have degree at least $N(N-1)/2$. Hence in the thermodynamic limit, when N is large, the pfaffian will vanish for any fixed value of k .

Consider now the general case, $\phi(z) = 1/z + p(z)$ with $p(z)$ of degree k . In expanding the pfaffian, one may for each pair choose either the $1/z$ or the $p(z)$. Suppose that the $1/z$ factor is chosen for r pairs, and $p(z)$ for the other $N/2 - r$. Fixing the r pairs and antisymmetrizing over the other variables, we see by the same argument as in the previous paragraph that the term will vanish unless $(N/2 - r)(N/2 - r - 1)/2 \leq k(N/2 - r)$, i.e. $N/2 - r - 1 \leq k$. Thus for fixed finite k one must choose the $1/z$ factor for almost all the pairs, that is all but $k + 1$. The resulting wave function is then most reasonably regarded as being generated from the state with $\phi(z) = 1/z$ by allowing a finite number of broken pairs, as we shall see in more detail below.

In the classification of superconducting pairing as s-wave, p-wave, ... the wave function based on (3.36) falls off the scale: it is a pairing in the $l = -1$ wave! That is, its angular momentum is unity, but instead of the radial wave function having a centrifugal barrier, it is actual enhanced near the origin. One may also construct states of the general form (3.33) at $\nu = 1/4, 1/6, \dots$ in the obvious way. (These filling fractions lie along the same anyon metal line, extrapolated increasingly further from free fermions.) For the suggested wave function at these filling fractions, higher-order poles in ϕ become allowed.

3.5. PFAFFIAN FACTS

The preceding considerations have led us to consider wave functions in which the mathematical objects known as pfaffians make a prominent appearance. Since these objects are probably not in the mathematical tool chest of most physicists, we shall collect here a few facts about them that will be useful in our subsequent considerations [28].

Given an antisymmetric matrix M_{ij} the *pfaffian* of M is defined to be

$$\text{Pf}(M) \equiv \sum_{\text{pairings}} (\pm) \prod_{\text{pairs}(ab)} M_{ab}, \tag{3.37}$$

where the sign associated with each term is positive if the permutation needed to bring the indices back to their original order is even, and negative if the required permutation is odd. Thus for example the pfaffian of a 4×4 matrix is

$$M_{12}M_{34} - M_{13}M_{24} + M_{14}M_{23}.$$

Perhaps the most important fact about the pfaffian is that its square is equal to the determinant,

$$\text{Pf}(M)^2 = \det(M). \tag{3.38}$$

For a symmetric matrix M_{ij} one defines the *haffnian*

$$\text{Hf}(M) = \sum_{\text{pairings}} \prod_{\text{pairs}(ab)} M_{(ab)}. \tag{3.39}$$

In the important special case $M_{ij} = (z_i - z_j)^{-2}$ (diagonal entries zero) one has the identity

$$\text{Hf}\left(\frac{1}{(z_i - z_j)^2}\right) = \det\left(\frac{1}{(z_i - z_j)}\right). \tag{3.40}$$

The identity (3.40) can be derived from the corresponding case of (3.38) (that is, with $M_{ij} = 1/(z_i - z_j)$) by an instructive argument, that will be used again in several related contexts below. To derive eq. (3.40) from (3.38) in this special case, it is clearly both necessary and sufficient to show that when one takes the square of the pfaffian all the cross terms vanish. To see this, consider the result of collecting all terms in which a fixed proper subset I of the variables z participate in double poles. They will form an expression

$$\prod_{i,j \in I} \frac{1}{(z_i - z_j)^2} \times P(z_k), \tag{3.41}$$

where P is a polynomial in the inverse differences of the remaining variables (those which do not belong to I). We claim $P = 0$. To see this, consider the result of multiplying P by the Vandermonde determinant in these variables:

$$\prod_{i < j} (z_i - z_j) = \det V_{ij}, \tag{3.42}$$

where $V_{ij} \equiv (z_i)^j$. The result of the multiplication is an antisymmetric polynomial of lower degree than the Vandermonde determinant. However, it is clear from the identity (3.42) that the Vandermonde determinant is the non-trivial antisymmetric polynomial of lowest possible order, since it has all the requisite zeroes as linear factors. Thus the result of the multiplication vanishes, and therefore P vanishes, as claimed. This proves that all the cross terms in the square of the pfaffian cancel, and therefore that eq. (3.40) follows from the corresponding case of (3.38).

3.6. A HAMILTONIAN, FOR WHICH THE WAVE FUNCTION IS EXACT

It is an extremely important fact that the pfaffian wave function is the exact ground state of a simple local effective hamiltonian. Without some such property, even the existence of a sensible thermodynamic limit would not be at all evident, nor even particularly likely.

The hamiltonian in question is inspired by the classic model hamiltonians for which the Laughlin $1/m$ states are exact eigenstates [29]. Because the wave function for the $1/m$ state contains the factor $(z_i - z_j)^m$ for each pair, it is annihilated by

$$V^{(m)} \equiv \sum_{i < j} V_{ij}^{(m)}, \tag{3.43}$$

where

$$V_{ij}^{(m)} = (\nabla_i^2)^{(m-1)/2} \delta^{(2)}(z_i - z_j), \tag{3.44}$$

and of course by the same thing with any smaller power of the laplacian. Thus the wave function is an exact eigenstate of the hamiltonian with short-range interparticle repulsions of this form and the usual kinetic term for charged particles in an external magnetic field. Note that the expectation value of the potential energy is positive definite for the sign corresponding to repulsion. While a pure δ -function repulsion is negligible for any non-singular fermion wave function, due to the antisymmetry of the wave function, the vanishing of these derivative interactions is a highly non-trivial property, and indeed the Laughlin $1/m$ states are non-degenerate.

(Actually there is a subtlety here, well-known to experts in the quantized Hall effect. One could include additional multiplicative polynomial factors on top of the

given wave function and still have an eigenstate with the same eigenvalue, formally. For instance the standard quasihole has this character. However these factors if added in a haphazard way will not lead to a wave function which has a sensible thermodynamic limit, or if added in a systematic way (as is done in the case of the quasihole) will change the density. Thus if a suitable pressure or chemical potential term is added, one will be driven back to the pure $1/m$ state as the ground state. Perhaps the simplest way to implement this is to add a term proportional to the total angular momentum, or equivalently the degree of the polynomial factor. The $1/m$ state is the antisymmetric polynomial of smallest degree that is annihilated by $V(m)$. Alternatively one may work on a sphere or torus to avoid boundary problems; but then one must add a quasiparticle together with the quasihole, and the energy of the pair is definitely higher than the energy of the ground state.)

These potentials will not quite work in our case. However, the pfaffian state (3.33) does have a very special property, which allows a simple modification of the construction to go through. The property in question is this: for any term in the wave function, and any particular coordinate z_i , there is at most one other z_j for which $z_i - z_j$ fails to appear at least quadratically. This property implies that the three body potential

$$V = \sum_{\text{triples}} V_{i;jk}, \quad (3.45)$$

with

$$V_{i;jk} = \nabla_i^2 \left(\delta^{(2)}(z_i - z_j) \delta^{(2)}(z_i - z_k) \right), \quad (3.46)$$

annihilates the pfaffian wave function (3.33). Furthermore V is manifestly positive definite, since in evaluating its expectation value one can integrate by parts to peel one derivative off to each side, thereby arriving at a sum of squares weighted by a positive measure.

We can also argue that the polynomial factor in $\Psi_{1/2}$ is the unique antisymmetric polynomial of lowest degree annihilated by V . Indeed, let P be such a polynomial, and let us consider the part of P in which a fixed product $\rho = \prod(z_a - z_b)$ of linear factors appears. (All other factors in $(z_i - z_j)$ are quadratic.) Thus, all the terms containing this product ρ , and no other linear difference factors, can be written as $\rho\lambda$. If there are l linear factors in ρ , then λ will be of degree $2 \times (\frac{1}{2}N(N-1) - l)$, at least. Now, in ρ a given z_a can appear at most once, for otherwise the term will not be annihilated by V . (There is no possibility for cancellations between terms with different choices of the linear factors, since such terms have different functional forms.) Thus $l \leq N/2$, and the inequality is saturated by choosing fixed pairs of the z 's and a linear difference factor for each pair. The lowest possible degree is thus attained if and only if $l = N/2$ and λ is

exactly quadratic in remaining difference factors. These requirements fix the form of the part of P under discussion to be

$$\prod_{\text{fixed pairs } z_a = z_b} \frac{1}{z_a - z_b} \prod_{i < j} (z_i - z_j)^2.$$

Now the second factor in this expression is symmetric. To form an antisymmetric polynomial incorporating the part of P under discussion we must antisymmetrize. But in so doing we arrive back precisely at $\Psi_{1/2}$.

Thus $\Psi_{1/2}$ has a similar status for the potential V as the Laughlin $1/m$ state has for $V^{(m)}$. By adding a pressure, a chemical potential, or an energy term proportional to the total angular momentum, we can insure that it is the unique ground state.

In a similar fashion, one may show that the higher analogs of the $\nu = 1/2$ state for $\nu = 1/4, 1/6, \dots$ are eigenstates including appropriate higher-gradient repulsions. Another notable case is the $\nu = 1$ *boson* state constructed by multiplying the standard fermion wave function for a full Landau level (involving a Vandermonde determinant) by the $1/z$ pfaffian. This is actually a particularly simple case, since the three-body hamiltonian can be taken to contain bare delta-functions, rather than their gradients. It is amusing that repulsive interactions can cause *bosons* to fill a Landau level essentially exactly, and to pair – a behavior usually associated with fermions.

4. Topological quantum numbers

A deep feature of the quantized Hall states is the topological character of their ordering. Their ordering apparently cannot be characterized by a local order parameter; instead, it is better captured by order parameters of a global or topological character [30]. Operationally, the relevant point for us here is that one can identify discrete integers which serve as signatures for the universality classes. If some integers characterize two states which are continuously related, then clearly the value of these integers for the two states must be equal. Thus, to test our claim that the state we reached by heuristic arguments based on adiabatic development from the BCS pairing state, which plausibly is well described by a pfaffian trial wave function, is in the same universality class as Halperin's strong-pairing state, we must verify that the characteristic integers for these states are equal.

The particular integers we shall discuss are the flux-particle number displacement and the ground state degeneracy on spheres and tori. The displacement is defined as follows. The incompressible ground state occurs when there is a particular relation between flux and particle number. In the thermodynamic limit,

this relation is of course simply $N_\phi = (1/\nu)N$, where N_ϕ is the number of magnetic flux quanta, N is the number of electrons, and ν is the filling fraction. For finite N , however, the relation may be different, and in fact will depend on the topology of the compact surface on which the system is defined. We shall find, for example, that on a sphere we have for the paired Hall state the extremely important relation

$$N_\phi = 2N - 3, \quad (4.1)$$

between number and flux.

The ground state on a torus will turn out to be 8-fold degenerate, a fact closely related to the anyon statistics of the quasiparticles.

Our other task in this section will be to argue that the pfaffian trial wave function is in some sense physically unique within the lowest Landau level, and is in a precise sense unique for a specially crafted, but quite simple, local hamiltonian.

4.1. EXPECTATIONS FROM THE STRONG PAIRING LIMIT

In the strong pairing limit for $\nu = 1/2$, we replace the fundamental $N = 2N_b$ fermions with N_b effective bosons of twice the charge. These bosons are then at filling fraction $\nu_b = 1/8$, which is an allowed value for a Laughlin state of bosons. To read off the topological numbers for the state, we can use the explicit form of the Laughlin wave functions on spheres and tori.

Let us briefly recall the formalism for charged particles subject to a uniform magnetic field on a sphere [6]. Thus, we are studying charged particles subject to the influence of a magnetic monopole source at the origin. For this purpose, it is convenient to introduce spinor co-ordinates u, v defined as

$$u = \cos(\theta/2) e^{i\phi/2}, \quad v = \sin(\theta/2) e^{-i\phi/2}, \quad (4.2)$$

where θ, ϕ are the polar coordinates on the unit sphere. (u, v of course form a double cover of the sphere, since they change sign under $\phi \rightarrow \phi + 2\pi$.) Let us assume that there are N_{ϕ_b} units of flux passing through the sphere, and choose a gauge such that $A = (N_{\phi_b}/qR) \cot(\theta)\hat{\phi}$.

The single particle wave functions for the lowest Landau level are the homogeneous polynomials of a fixed total degree d in u, v . Thus the $d + 1$ monomials $u^d, u^{d-1}v, \dots, v^d$ form a basis. They fill out a representation of spin $d/2$. For particles with no intrinsic spin, these functions define the lowest Landau level when the flux through the sphere is $(d/2)(4\pi/q) \equiv d\Phi_q$, where q is the charge of the particle and Φ_q the associated flux quantum. When the particles have spin s , the relation is changed in a very simple way, because the coupling of spin to curvature in this geometry is just like the coupling of charge to magnetic flux. Thus, the effect of

spin is to shift the degree of the allowed polynomials by $-2s$, for a given value of flux.

What should we take for the spin of the pair? Clearly (since the components are identical fermions) it must be an odd integer. In principle any odd integer s might occur for a suitable potential. Considerations such as those in the previous section, however, strongly recommend $s = -1$ to our attention. Indeed, it is only this partial wave which lowers the degree of the zero as two particles approach – and thus corresponds to pair formation – without introducing a singularity.

Now in line with the prescription $z \rightarrow u/v$ the form of the Laughlin factor for the droplet

$$\Psi_{\text{droplet}} = \prod_{i < j} (z_i - z_j)^8 \tag{4.3}$$

goes over into

$$\Psi_{\text{sphere}} = \prod_{i < j} (u_i v_j - v_i u_j)^8 \tag{4.4}$$

on a sphere. Note that no additional gaussian localizing factor is required. Ψ_{sphere} , taken literally, is the canonical Laughlin trial wave function on a sphere. It is demonstrably the non-degenerate ground state for simple trial potentials – see below. Ψ_{sphere} is homogeneous in the two variables u_i, v_i for any i , of degree $8(N_b - 1)$. Thus it represents a wave function in the lowest Landau level for flux and spin

$$N_{\phi_b} + 2s = 8(N_b - 1). \tag{4.5}$$

Now with $N_{\phi_b} = 2N_\phi, N_b = \frac{1}{2}N$ this becomes the number–flux relation

$$N_\phi = 2N_f - 3, \tag{4.6}$$

as anticipated above.

Let us now briefly recall the formalism for construction of the corresponding states on a torus [31]. We identify functions on the torus with functions in the complex plane periodic in 1 and τ . We suppose that N_{ϕ_b} units of flux penetrate the torus, and adopt the symmetric gauge

$$\mathbf{A}(\mathbf{r}) = \frac{1}{2}B(\mathbf{r} \times \hat{z}). \tag{4.7}$$

In this gauge, the single particle wave functions in the lowest Landau level take the form

$$\psi(z) = \exp\left(-\frac{eB}{4}(|z|^2 - z^2)\right) f(z), \tag{4.8}$$

where $f(z)$ is an arbitrary holomorphic function on the torus, that is, periodic with periods 1, τ . Equivalently, ψ is a holomorphic function in the complex plane subject to the quasiperiodicity conditions

$$\psi(z + \xi) = \exp\left(-\frac{eB}{4}(\bar{z}\xi - z\bar{\xi})\right)\psi(z), \tag{4.9}$$

for $\xi = 1, \tau$.

In constructing explicit wave functions, the ϑ functions [32] are indispensable. They are defined by

$$\vartheta_1(z, \tau) = \sum_{n=-\infty}^{+\infty} \exp\left(\pi i\left(n + \frac{1}{2}\right)^2 \tau\right) \exp\left(2\pi i\left(n + \frac{1}{2}\right)\left(z + \frac{1}{2}\right)\right), \tag{4.10}$$

$$\vartheta_2(z, \tau) = \vartheta_1\left(z + \frac{1}{2}, \tau\right),$$

$$\vartheta_3(z, \tau) = M\vartheta_1\left(z + \frac{1}{2}(1 + \tau), \tau\right),$$

$$\vartheta_4(z, \tau) = M\vartheta_1\left(z + \frac{1}{2}\tau, \tau\right). \tag{4.11}$$

where $M \equiv e^{i\pi\tau/4} e^{i\pi z}$. For our purposes the most important properties of the ϑ functions are the periodicities

$$\begin{aligned} \vartheta_1(z + 1) &= -\vartheta_1(z), & \vartheta_2(z + 1) &= -\vartheta_2(z), \\ \vartheta_3(z + 1) &= +\vartheta_3(z), & \vartheta_4(z + 1) &= +\vartheta_4(z), \end{aligned} \tag{4.12}$$

$$\begin{aligned} \vartheta_1(z + \tau) &= -e^{-i\pi\tau} e^{-i2\pi z} \vartheta_1(z), & \vartheta_2(z + \tau) &= +e^{-i\pi\tau} e^{-i2\pi z} \vartheta_2(z), \\ \vartheta_3(z + \tau) &= -e^{-i\pi\tau} e^{-i2\pi z} \vartheta_3(z), & \vartheta_4(z + \tau) &= -e^{-i\pi\tau} e^{-i2\pi z} \vartheta_4(z), \end{aligned} \tag{4.13}$$

the reflections relations

$$\begin{aligned} \vartheta_1(-z) &= -\vartheta_1(z), & \vartheta_2(-z) &= \vartheta_2(z), \\ \vartheta_3(-z) &= \vartheta_3(z), & \vartheta_4(-z) &= \vartheta_4(z), \end{aligned} \tag{4.14}$$

and the fact that ϑ_1 is a holomorphic function whose only zeroes are simple ones occuring at the lattice points $m + n\tau$, where m, n are integers. In the following, we shall often leave τ as an implicit parameter, as we have already done in (4.14).

After these preliminaries it is possible to construct the appropriate generalization of the Laughlin wave function on the torus, as follows. One factorizes the wave function as a product of center-of-mass and relative coordinate pieces, in the

form $\Psi_{\text{torus}} \equiv \psi_{\text{c.m.}} \psi_{\text{rel.}}$. The relative coordinate piece is simply

$$\psi_{\text{rel.}} = \prod_{i < j}^N \vartheta_1(z_i - z_j | \tau)^8 \tag{4.15}$$

The center-of-mass piece is more intricate. One can choose

$$\psi_{\text{c.m.}} = \prod_{i=1}^N \exp\left(-\frac{eB}{4}(|z_i|^2 - z_i^2)\right) \exp(iKZ) \prod_{\nu=1}^8 \vartheta_1(Z - Z_\nu | \tau), \tag{4.16}$$

where $Z = \sum z_i$ is the center-of-mass coordinate. The real parameter K and the center-of-mass zeros Z_λ may be chosen arbitrarily subject to the constraints

$$\begin{aligned} (-1)^{N_\phi} \exp(iK) &= 1, \\ (-1)^{N_\phi} \exp(iK\tau) \exp\left(2\pi i \sum_\nu Z_\nu\right) &= 1. \end{aligned} \tag{4.17}$$

It may be verified that the wave function thus constructed is indeed of the required form (4.8) in each variable separately. In the process of this verification one determines the number–flux relation, which is required to be $N_{\phi_b} = 8N_b$, or

$$N_\phi = 2N_f. \tag{4.18}$$

The fact that the center-of-mass wave function contains arbitrary parameters indicates a degeneracy of the ground state. How large is this degeneracy? It may be shown that it is (for the bosonic $\nu = 1/8$ state presently under discussion) 8-fold degenerate. Indeed the defining properties we require of the center-of-mass part $\psi_{\text{c.m.}}$ of the wave function, are that it is holomorphic, that it has exactly m zeroes in the principal region, and that it satisfies the appropriate boundary condition, which is linear in $\psi_{\text{c.m.}}$. Thus given one solution $\psi_{\text{c.m.}}$, its ratio $\tilde{\psi}_{\text{c.m.}}/\psi_{\text{c.m.}}$ with any other solution $\tilde{\psi}_{\text{c.m.}}$ is a meromorphic truly periodic function on the torus, with at most simple poles at 8 prescribed points (namely the zeroes of $\psi_{\text{c.m.}}$). It is a standard theorem in complex function theory – a very special case of the Riemann–Roch theorem – that the space of such functions is 8-dimensional, including the constant function [33]. Consequently, a Laughlin 1/8-state subject to periodic boundary conditions is 8-fold degenerate.

The degeneracy has an appealing intuitive explanation in terms of the anyon statistics of the quasiparticles. These quasiparticles are produced by adiabatic insertion of a single flux unit, and are localized excitations with no internal coordinates. They carry fractional charge 1/8 the charge of the effective pair bosons, and obey anyon statistics with parameter $\pi/8$. Now consider the process displayed in fig. 1, where the particles return to their original positions after

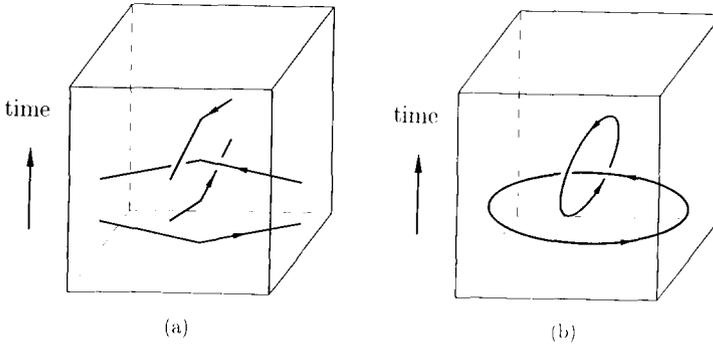


Fig. 1. The sequence of four processes displayed in (a) – creation of particle hole pairs, adiabatic transports around one of the two meridians of the torus, and subsequent annihilations of the pairs – is topologically equivalent to entangling the world lines of quasiparticles, as shown in (b).

looping the torus to-and-fro, but their world lines are entangled. According to the basic definition of anyons, this process should be associated with the phase factor $e^{i2\pi/8}$. Thus denoting the unitarity operators which implement the transition between states related by adiabatic transport of a particle around the two meridians by T_1, T_2 we have the commutator relation

$$T_1 T_2 T_1^{-1} T_2^{-1} = e^{2\pi i/8}. \quad (4.19)$$

This form of commutator often arises in problems involving translation in a magnetic field. It is well known that the smallest space on which it can be implemented is 8 dimensional; indeed the canonical realization of the algebra is in the form

$$T_1 \mathbf{v}_n = \exp\left(\frac{2\pi i}{8} n\right) \mathbf{v}_n \quad (4.20)$$

$$T_2 \mathbf{v}_n = \mathbf{v}_{n+1}$$

where the \mathbf{v}_n are eight orthonormal basis states, and the subscripts are to be interpreted as integers modulo 8. Now if inserting a flux tube is a unique operation and the quasiparticles have no internal quantum numbers, the degeneracy of the state with quasiparticles must reflect a pre-existing degeneracy of the ground state. Thus we see that the 8-fold degeneracy, which appeared as a rather complicated technical by-product in the construction of explicit wave functions, has a profound physical basis.

Let us summarize this discussion of the topological numbers we infer in the strong pairing limit. On the sphere we find a non-degenerate ground state and the flux-number relation $N_\phi = 2N - 3$ (for pair spin $s = -1$). On the torus we find an

8-fold degenerate ground state and the flux–number relation $N_\phi = 2N$, independent of the pair spin.

4.2. THE WAVE FUNCTION OF THE SPHERE: FLUX SHIFT

It is straightforward to construct the appropriate generalization of the pfaffian wave function to the sphere. Thus the suggested trial wave function for the paired Hall state on a sphere is $\Psi_{\text{sphere}} = \psi_1 \psi_2$ where

$$\psi_1 = \text{Pf} \left(\frac{1}{u_i v_j - v_i u_j} \right), \tag{4.21}$$

$$\psi_2 = \prod_{i < j} (u_i v_j - v_i u_j)^2. \tag{4.22}$$

The total wave function is homogeneous in each pair u_i, v_i of degree $-1 + 2(N - 1) = 2N - 3$. Thus we have the advertised number–flux relation (4.1) directly, in agreement with the strong-pairing limit (for $s = -1$).

4.3. THE WAVE FUNCTION ON THE TORUS: DEGENERACY

To construct the generalization of the pairing wave function for a torus, one additional ingredient is required, beyond what was necessary for the traditional Laughlin wave function. That is, of course, a prescription for generalizing the pfaffian.

The i, j element of the pfaffian certainly must include a factor $1/\vartheta_1(z_i - z_j)$, which is the appropriate semi-naive generalization of $1/(z_i - z_j)$. However the inverse ϑ function by itself will not do, because it wrecks the delicate periodicity relations for the overall wave function. Fortunately there is a simple remedy: one may include any of the *even* theta functions in the numerator. Thus there are three degenerate states where the i, j element of the pfaffian is

$$M_{ij} = \frac{\vartheta_a(z_i - z_j)}{\vartheta_1^2(z_i - z_j)}, \tag{4.23}$$

with $a = 2, 3, 4$ respectively. Since this is a ratio of theta functions with the same argument, it behaves simply under translation through the periods of the torus (in particular, introducing no z -dependent factors). Therefore the previous construction for the Laughlin state may be used, with this pfaffian inserted as a prefactor, to construct the pairing Hall state on a torus.

Much experimentation has convinced us that there are no further possibilities for the pfaffian along these lines, and we have actually proved it for the case of two

particles. Thus it would appear that the pfaffian factor introduces a degeneracy of three for the ground state. (This factor three would also be anticipated if one used the conformal field theory of the Ising model to construct the state; in this context it is the statement that there are three conformal blocks.) There is an additional, independent, two-fold degeneracy associated with the center-of-mass zeroes. Thus the total degeneracy, for the three-body hamiltonian model introduced previously, is six.

On the other hand in the strong pairing limit we found an eight-fold degeneracy. The physical arguments presented above, and the numerical evidence to be presented below, indicated that we should therefore expect this degeneracy also for the exact model. And we shall see below, that in the presence of two separated quasiparticles there is an exact eight-fold degeneracy for wave functions built on the pfaffian ground state. Given this result, it would require internal quantum numbers and/or peculiar long-range interactions between the quasiparticles for the ground state to have a different degeneracy from that of the ground state. Both these arguments, suggest the physical necessity of an eight-fold degeneracy for the ground state. What's going on?

We would like to suggest that the discrepancy can be reconciled in the following way. While it is true that the *exact* ground state is only six-fold degenerate, if the state is incompressible there must be two additional states which are separated from the ground state by an energy gap which vanishes in the thermodynamic limit.

In favor of this point of view, we would like to compare a similar situation in the theory of ordinary superconductors, and point to the importance of \mathbb{Z}_2 gauge structures in both contexts ^{*}.

As is of course well known, because of pairing the flux unit for an ordinary superconductor is $h/2e$ rather than h/e . On the other hand, single-electron wave functions around a loop enclosing an odd integer multiple of flux $h/2e$ are not single valued; rather, they change sign upon transport around the loop. The point is that the order parameter, which must be single-valued around the loop, is quadratic in the single-electron wave function. More abstractly, the point is that the superconducting state only requires long-range correlations in wave functions invariant under a residual \mathbb{Z}_2 gauge group, the unbroken remanant of the original U(1) electromagnetic gauge symmetry. It is useful, in constructing explicit trial wave functions for the superconducting state, to write them as sums of products of single-particle wave functions, but these single-particle wave functions are not to be taken too literally, because the only long-range correlations in the system are typically correlations among different pairs. In particular, single particle wave functions are not correlated with themselves over long distances. Therefore they may be antiperiodic around closed loops – indeed, it is *necessary* to allow them to be, in order to reproduce the experimentally observed flux quantum $h/2e$.

^{*} The following discussion is closely related to the classic treatment of Byers and Yang [34].

To be more concrete, consider the classic situation of a thick cylinder threaded by flux $\Phi = s(h/e)$, and let us discuss the angular dependence of the wave functions in the regular gauge $A_\phi = (1/2\pi)s(h/e)$. Suppose that the pairing for zero flux occurs between states of opposite angular momentum $\pm l$, that is wave functions $\psi(\phi) \propto e^{\pm il\phi}$. Then by an adiabatic argument the pairing would like to occur between states with wave functions $e^{i(\pm l+s)\phi}$ when the flux is inserted. In general, of course, these wave functions are not single valued. The multi-valuedness, as such, is a red herring (i.e. at the one-particle level), because all gauge invariant quantities constructed from ψ are single valued, and these are generally all that are physically meaningful. However, in the superconducting phase we require coherence in the condensate, which breaks the gauge symmetry. This coherence, captured in the order parameter, must be reflected in the wave function constructed from the product wave functions for pairs. Thus we must require, in order to construct the appropriate correlated wave functions, that the products

$$\psi_{\text{pair}}(\phi_1, \phi_2) \propto e^{il(\phi_1 - \phi_2)} e^{is(\phi_1 + \phi_2)} - (1 \leftrightarrow 2), \quad (4.24)$$

are single valued *with respect to the pair coordinate*, that is when both ϕ_1 and ϕ_2 are translated through 2π . (Here and below, in the interests of simplicity we have ignored the spin degree of freedom; thus we impose antisymmetry on the spatial wave function. There is no essential difficulty in incorporating spin.) Clearly, for this, it is enough that s is half of an integer. If s is half of an odd integer, then the one-particle wave functions will not be single valued; rather they will be antiperiodic under $\phi \rightarrow \phi + 2\pi$.

Of course if we wanted to construct a state where there was condensation in the one-particle sector, then we would have to demand that the wave functions in the one-particle sector are single valued, and s would be required to be an integer.

For our purposes, it is especially important to re-consider the preceding argument in real space, as opposed to (angular) momentum space. This is actually a more profound, and at the same time more straightforward way of dealing with the problem. (Though it may be less familiar – we have not found it in precisely this form in the published literature). The real-space form of the BCS trial wave function is the pfaffian

$$\Psi(\phi_i) = \text{Pf}(\psi_{\text{pair}}(\phi_i, \phi_j)) \quad (4.25)$$

where ψ_{pair} is the wave function of a pair. According to the previous arguments, if half an odd integer unit of flux threads the cylinder, the pair wave function must obey the boundary conditions

$$\begin{aligned} \tilde{\psi}_{\text{pair}}(\phi_1 + 2\pi, \phi_2) &= -\tilde{\psi}_{\text{pair}}(\phi_1, \phi_2), \\ \tilde{\psi}_{\text{pair}}(\phi_1, \phi_2 + 2\pi) &= -\tilde{\psi}_{\text{pair}}(\phi_1, \phi_2). \end{aligned} \quad (4.26)$$

Now we would like to argue that this boundary condition makes little difference to the bulk state, if the pair wave functions are short-range. Indeed, if the ordinary pair wave function (without the flux-modified boundary conditions) satisfies the locality requirement

$$\psi_{\text{pair}}(\phi_1, \phi_2) = 0 \quad \text{unless} \quad |[(\phi_1 - \phi_2)/2\pi]| < 1/2, \quad (4.27)$$

then one could simply define

$$\begin{aligned} \tilde{\psi}_{\text{pair}}(\phi_1, \phi_2) \\ \equiv (-1)^{\lfloor \phi_1/2\pi \rfloor} (-1)^{\lfloor \phi_2/2\pi \rfloor} \psi_{\text{pair}}(\phi_1 - 2\pi \lfloor \phi_1/2\pi \rfloor, \phi_2 - 2\pi \lfloor \phi_2/2\pi \rfloor), \end{aligned} \quad (4.28)$$

where $\lfloor x \rfloor$ is, by definition, the greatest integer less than or equal to x . Because of the locality requirement (4.27), the pair wave function defined this way does not have any discontinuities, and it obviously resembles ψ locally (and leads to a state with equal energy) while obeying the boundary condition (4.26).

Now the true pair wave function will not quite obey the strict locality requirement (4.27), but if we are dealing with a macroscopic object it will be true that the wave function of the pair will become very small when the members of the pair are on opposite sides of the sample, and this requirement will be satisfied to an excellent approximation. (In real superconductors, the pair size is given by the coherence length, which is typically of order a micron or less.) The perturbation of the pair wave function necessary to implement the flux boundary condition (4.26) will not be strictly trivial in the above sense, but it will be vanishingly small as the sample becomes large, and the change in the bulk energy density will vanish in the thermodynamic limit. On the other hand if the pair wave function is itself long-range, this argument will fail. This must happen in particular if there is substantial amplitude for a single particle to loop the loop coherently. In such a case, there will be strict periodicity only in the larger flux unit h/e .

Thus, to summarize, the halving of the flux quantum for a paired superconductor not only is consistent with, but is in a precise sense equivalent to, the use of non-trivial \mathbb{Z}_2 gauge structures for the single-particle wave functions. If the pairing wave functions are local, these different gauge structures yield states with the same bulk energy density in the thermodynamic limit.

Now let us reconsider the pair wave functions we have constructed in connection with the paired Hall state from this point of view. Under translations around the cycles of the torus, i.e. translations through 1 or τ , the ratios $\vartheta_2(z)/\vartheta_1(z)$, $\vartheta_3(z)/\vartheta_1(z)$, $\vartheta_4(z)/\vartheta_1(z)$ are multiplied respectively by the factors $(+, -)$, $(-, -)$, and $(-, +)$. Thus, they give implementations of 3 out of 4 of the possible \mathbb{Z}_2 structures. Clearly, the missing element is the $(+, +)$ sector. It is a simple mathematical fact that there does not exist an analytic function of the kind we

would want for the pair wave sector in this sector, that is, truly doubly periodic with a single isolated zero. However, as we have argued above, insofar as the physical pair wave function may be taken to be short-ranged, the different sectors can be implemented with nearly identical energies. Now in the Hall state we expect that correlations in a pair that is separated by many magnetic lengths cannot be energetically significant. Thus, we expect all four sectors to be degenerate in the thermodynamic limit. Concretely, we expect that good trial wave functions can be constructed by applying procedures similar to (4.28) to any of the pair wave functions above, with a suitable cutoff.

From this perspective, it appears that the difficulty in finding the physically necessary fourth state stems from the fact that analytic functions are so rigid. Thus strictly confining oneself to the lowest Landau level makes smoothing constructions, that manifestly do not cost significant energy, appear to be impossible. If we are correct, then by including a small admixture of higher Landau levels, which costs arbitrarily little in the thermodynamic limit, one can gain all the correlation energy of the pairing in any flux sector.

The arguments of the preceding paragraphs are certainly not mathematically rigorous, but we find their internal coherence compelling. Thus we conclude that an incompressible state of the pairing type, whose ground state wave function resembles the pfaffian form, will have an eight-fold degenerate ground state on the torus in the thermodynamic limit. Whether the very special three-body hamiltonian gives actually rise to an incompressible state is not clear to us, but an incompressible state is a generic case. If the paired state for this special hamiltonian is incompressible, then the six-fold degeneracy, and associated non-abelian statistics for the quasiparticles, will only hold for adiabatic transport with frequency (times \hbar) smaller than the tiny gap to the two extra states. The qualitative physics of the strong-pairing limit, including abelian anyon statistics for the quasiparticles, will occur for frequencies larger than this gap but smaller than the gap to true quasiparticles.

5. Charged excitations: halving the flux

5.1. QUANTUM NUMBERS OF HALBERONS

It is easy to determine the quantum numbers of charged quasiparticles in the strong pairing limit. Indeed in that limit the physics is essentially that of an ordinary Laughlin state, with the unusual features that the fundamental particles are charge $2e$ bosons. The quasiparticles (or, more precisely, the quasiholes) may be created by the adiabatic insertion of magnetic flux pointing in the same direction as the uniform background magnetic field and localized at a point. The inserted flux is slowly increased from zero until there is one full unit of quantized

flux, i.e. in this context flux $h/2e$. At the conclusion of this adiabatic process, the flux will be essentially unphysical – it may be removed by a gauge transformation. (The gauge transformation is singular at the insertion point, but the wave function vanishes there.) Thus the notional external field used in the construction need not correspond to anything real. Nevertheless the adiabatic process of turning it on produces a local density deficit, as electrons acquire increased angular velocity from torques arising from the induced electric fields. Standard arguments allow one to infer that the charge of the quasihole is $-1/8$ the charge of the fundamental particles – that is $-e/4$ – and that they are anyons with statistical parameter $\theta/\pi = 1/8$.

While the insertion of quantized flux $h/2e$ is of course extremely natural from the point of view of the strong pairs, it is unusual from the point of view of the electrons. Likewise the charge and statistics of the quasiholes, which are respectively $1/2$ and $(1/2)^2$ of what would arise in a naive extrapolation from the Laughlin $1/m$ states to $m = 2$. Since the minimal charged excitations are, roughly speaking, half of what might have been expected, we call them *halberons*, after the German “halb”, meaning half.

On general principles we may expect that halberons, with the quantum numbers just derived, will exist and will be the minimal charged excitations, at least for small perturbations of the strong pairing limit. Realistic paired Hall states may be rather far from this limit, though in the same universality class. One may still draw the general conclusion that there are physical states with finite energy having the exotic quantum numbers of halberons, but whether the basic charged excitations have these quantum numbers is a quantitative question of energetics. For example, it is conceivable, at this level, that for suitable potential a paired Hall state forms in which the energy of two widely separated halberons is much larger than the energy of a localized charge $-e/2$ excitation, though we do not expect this to occur for basically repulsive potentials (which disfavor large density contrasts).

5.2. LIMITATIONS OF THE FLUX INSERTION PARADIGM

Once we step outside the strong pairing limit, it becomes clear that the simple flux insertion paradigm for construction of the halberons will no longer work. The notional point flux of strength $h/2e$ can certainly *not* be gauged away. Indeed, as we have discussed in sect. 4, such a flux will alter the boundary conditions on the fermion wave functions. Just as insertion of a full $h/2e$ at the origin leads to a factor $\prod_i z_i$ in the wave function, insertion of $h/2e$ flux at the origin into an ordinary or paired Hall state would lead to a $\prod_i \sqrt{z_i}$ factor in the wave function, which in the absence of a genuine magnetic field is certainly unacceptable.

(At the risk of belaboring the obvious, to avoid confusion let us mention explicitly that the process we are consider here, of insertion of flux into the body of the fluid, is quite distinct from the one we considered at length in sect. 4. There we

considered insertion of flux “through the holes” for an annular or toroidal geometry. The two are not entirely unrelated, of course. In either case, the energy density in the bulk – far from the place where the flux is inserted – is indetical, and it must be zero for values of the flux which are allowed with total finite energy in the thermodynamic limit. Thus the quantization rules for finite energy are the same in both cases.)

In fact we would like to mimic the effect of the real flux by particle correlations, without actually having the flux. Thus the angular momentum of each pair should be boosted by unity. However this is to be accomplished not by boosting the angular momentum of each component of the pair by $\frac{1}{2}$ – which would lead to the unacceptable square roots – but rather by boosting the angular momentum of one component of the pair by 1 and while leaving that of the other unchanged. The pairing of partial waves with angular momenta $(l, -l-1)$ for our effective p-wave superconductor is then changed to pairing of $(l, -l)$. This is accomplished very simply, by modifying the argument of the pfaffian factor according to

$$\frac{1}{z_i - z_j} \rightarrow \frac{z_i + z_j}{z_i - z_j}. \tag{5.1}$$

Indeed, for the angular dependence we then have

$$\frac{1}{z_i - z_j} = \frac{1}{z_i} \sum_{l=0}^{\infty} \left(\frac{z_j}{z_i}\right)^l \sim \sum_{l=0}^{\infty} e^{i(-l-1)\phi_i} e^{il\phi_j}, \tag{5.2}$$

going over into

$$\frac{z_i}{z_i - z_j} \sim \sum_{l=0}^{\infty} e^{i(-l)\phi_i} e^{il\phi_j}. \tag{5.3}$$

Note that these same partial waves are determined by starting with the pattern $(l, -l-1)$ for half-odd integer l , as is appropriate for p-wave pairing of particles subject to the boundary conditions for half a flux unit, and adiabatically adding a cancelling half flux unit, thus arriving at $(l + \frac{1}{2}, -l - \frac{1}{2})$.

The substitution (5.1) is also suggested directly by extrapolation from the strong pairing limit. One simply takes one additional power of the coordinate of the pair, where realized as simply the center-of-mass coordinate for the constituents of the pair.

5.3. WAVE FUNCTIONS FOR HALBERON PAIRS

It is instructive, and will prove useful, to consider halberons in spherical and toroidal geometries. In doing this, however, we must certainly respect the Dirac

quantization condition for the total fluxes flowing through the surfaces. Thus we must consider adding *pairs* of halberons to the ground state.

Let us begin, however, with the droplet. Let η and ζ be two fixed positions, where the half-fluxons will be localized. Then the first guess at a wave function for the pair might be to modify the pfaffian according to

$$\frac{1}{z_i - z_j} \rightarrow \frac{1}{z_i - z_j} \left(\frac{z_i + z_j}{2} - \eta \right) \left(\frac{z_i + z_j}{2} - \zeta \right). \quad (5.4)$$

However, with some hindsight from the generalization to sphere and torus, it seems preferable to delete the quadratic factors in z_a and z_b – which actually correspond to an edge excitation. Then we arrive at the pfaffian factor

$$\text{Pf} \left(\frac{1}{z_i - z_j} \right) \rightarrow \text{Pf} \left(\frac{(z_i - \eta)(z_j - \zeta) + (i \leftrightarrow j)}{z_i - z_j} \right). \quad (5.5)$$

Indeed when $\zeta \rightarrow \eta$ we should expect the two-halberon wave function to approach that of a single flux insertion, which is true for (5.5) but not for (5.4).

The generalization of (5.5) to a sphere is immediate. Using the substitution $z \rightarrow u/v$ and clearing fractions, we find

$$\text{Pf} \left(\frac{1}{u_i v_j - u_j v_i} \right) \rightarrow \text{Pf} \left(\frac{(u_i \beta_\eta - v_i \alpha_\eta)(u_j \beta_\zeta - v_j \alpha_\zeta) + (i \leftrightarrow j)}{u_i v_j - u_j v_i} \right), \quad (5.6)$$

where of course $(\alpha_\eta, \beta_\eta)$, $(\alpha_\zeta, \beta_\zeta)$ are the spinor co-ordinates of the halberons. Notice that this transcription of (5.5) is homogeneous of degree unity in each particle coordinate; thus it represents an acceptable multiplicative factor for a wave function on the sphere, and corresponds to one extra unit of flux (h/e) through the sphere. Transcription of (5.4) would not give an acceptable wave function on the sphere.

5.4. HALBERON PAIRS ON THE TORUS

The generalization to the torus is less straightforward, but very instructive. The most immediate generalization of (5.5) that suggests itself is perhaps to modify the pfaffian factor

$$\frac{\vartheta_a(z_i - z_j)}{\vartheta_1(z_i - z_j)} \rightarrow \frac{\vartheta_a(z_i - z_j)}{\vartheta_1(z_i - z_j)} (\vartheta_1(z_i - \eta) \vartheta_1(z_j - \zeta) + (i \leftrightarrow j)), \quad (5.7)$$

but this is inadequate to produce a well-defined overall wave function on the torus. The difficulty is that translation of (say) z_i through τ produces different phase

factors in each of the two terms of (5.7). On thorough reflection, one comes to realize that the correct generalization is given by

$$\text{Pf} \left(\frac{\vartheta_a(z_i - z_j)}{\vartheta_1(z_i - z_j)} \right) \rightarrow \text{Pf} \left(\frac{\vartheta_a(z_i - z_j + \frac{1}{2}(\eta - \zeta)) \vartheta_1(z_i - \eta) \vartheta_1(z_j - \zeta) + (i \leftrightarrow j)}{\vartheta_1(z_i - z_j)} \right), \tag{5.8}$$

and that only the center-of-mass coordinate of the halberon pair enters the center-of-mass part of the electrons

$$Z \rightarrow Z + \frac{1}{2} \frac{\eta + \zeta}{2}. \tag{5.9}$$

A particularly interesting feature of (5.8) is the factor $\frac{1}{2}$ that appears multiplying the halberon coordinates. It means that when this coordinate is transported around a cycle of the torus, the wave function will not come back to itself. This is certainly broadly consistent with the idea that the halberons are anyons, according to the ideas of Einarsson [35] and others [36]. Although we will not pursue this aspect of the subject in detail here, we do wish to point out its connection to our previous discussion of ground state degeneracy. For two halberons in general position, (5.8) actually represents four distinct states on the torus. These are labeled by the subscript a , which may be 1, 2, 3 or 4. By translating η (or ζ) through the periods of the torus, we transform these distinct states transform into each other, as can be seen easily from the definitions (4.11). Given this result, we might try to generate ground state configurations by allowing η to approach ζ , and then removing the excess flux h/e . Attempting this, we find that when the subscript is equal to 2, 3, or 4 we arrive exactly at the three degenerate states (4.23). However when the subscript a is equal to 1, the limiting wave function vanishes. This is a very peculiar fact, since (as we mentioned above) the different possibilities arise from each other by transporting a halberon around a cycle. At present we do not understand this very peculiar collapse of the wave function in physical terms. However, the fact that in the presence of two halberons the generic degeneracy is 4-fold (or 8-fold, taking account of the center of mass) is only consistent with the notion that the halberons are unique, localized excitations if the underlying ground state is equally degenerate.

6. Pair breaking excitations

6.1. A STATISTICAL PARADOX RESOLVED

The traditional quantized Hall states occur for odd denominator filling fractions. Tao and Wu [19] presented a general argument, which gave a fundamental

explanation of this fact. Since we are claiming the possibility of incompressible quantized Hall states at $\nu = 1/2$, an even denominator, we must address the question, how the argument of Tao and Wu is evaded.

Adapted to the case at hand, their argument runs basically as follows. The removal of an electron into the fluid must result in the creation of several quasiholes. Thus, it must be possible to produce a state of several quasiholes, which has the same quantum numbers as a single electron. Let us see if this is possible.

In the interest of simplicity let us adopt the convenient fiction that the minimal charged quasiparticles are di-halberons; we leave it as an exercise for readers to convince themselves that the conclusion is not changed by taking the existence of single halberons into account. Di-halberons are created by the normal flux insertion process, and are charge $-e/2$ statistical parameter $\theta/\pi = 1/2$ anyons. By way of comparison, the quasiholes of the traditional $\nu = 1/3$ state are charge $-e/3$ statistical parameter $\theta/\pi = 1/3$ anyons. Now when one puts together n anyons with statistical parameter α , the resulting composite has statistical parameter $n^2\alpha$ – there are n times as many particles, each acquiring n times as much phase, as one composite winds around the other. From this simple fact, we see that the $\nu = 1/2$ and $\nu = 1/3$ cases are profoundly different. In the former case, we reproduce the charge of an electron hole by forming a composite of two di-halberons. This composite has statistical parameter $\theta/\pi = 2^2 \times \frac{1}{2} = 2$ – it is a *boson*. By contrast in the latter case the charge of the electron hole is reproduced by a three-quasihole composite, with statistical parameter $\theta/\pi = 3^2 \times \frac{1}{3} = 3$ – a *fermion*. Clearly in the latter case, but not in the former, the quantum numbers match that of an electron hole. From this mismatch, Tao and Wu draw the conclusion that $\nu = 1/2$ is impossible.

By the way, of course, this argument would not produce any objection to the filling fraction $\nu = 1/8$ state of the effective bosons. Here the fundamental boson hole has the charge of 8 quasiholes with statistical parameter $\theta/\pi = 1/8$; and the 8-quasihole composite indeed has Bose statistics ($\theta/\pi = 8^2 \times \frac{1}{8} = 8$).

However, there is another way to interpret the argument. What the argument really shows, is that the traditional charged quasiparticles alone are not adequate to produce an electron hole. Thus, an incompressible $\nu = 1/2$ Hall fluid must contain additional quasiparticles. The simplest possibility to make up the difference between a two di-halberon composite and an electron hole, is a neutral fermion.

That the paired Hall state should support neutral fermion excitations, can also be argued in another way, at least heuristically. In a normal BCS superconductor, the pair-breaking excitations are neutral fermions. The neutral fermions of the paired Hall state, are just the adiabatic continuation along the anyon metal line of the pair-breaking excitations that exist in the mother superconducting state. To make this into a real argument, we would have to convince ourselves that neither

the charge nor the statistics of these excitations is altered in the process of continuation in quantum statistics. (This is not obvious – in fact, it is definitely false for the charged excitations). Without pretense of rigor, let us merely remark that the fact that the pair-breaking excitations are electrically neutral makes it quite plausible that the manipulation of magnetic fields involved in implementing the adiabatic heuristic does not change the nature of the interactions of these excitations with electrons, nor with each other. That is enough, to insure that neither the charge nor the statistics of the pair-breaking excitations change as one moves from the mother BCS superconductor to the paired Hall state.

There is an instructive consistency in the interpretation of the neutral fermions, whose existence was required to resolve a statistical paradox in the representation of electron holes, with pair-breaking excitations. For both the necessity of the neutral fermions, and the possibility of pair-breaking excitations, vanish in the strong-pairing limit. As we approach this limit, the energy cost for producing pair-breaking excitations escalates, and they become dynamically insignificant.

6.2. TRIAL WAVE FUNCTIONS

Now we will attempt a more concrete discussion, featuring specific trial wave functions.

A first pass at the wave function for the pair breaker is suggested by the preceding argument. It is given (for the droplet) by

$$\Psi_\eta = \mathcal{N} \exp\left(-\frac{eB}{4}(|z_1|^2 + |\eta|^2 - 2z_1\eta^*)\right) \prod_{i=2}^N (\eta - z_i)^2 \Psi_{1/2}(z_2, \dots, z_N), \tag{6.1}$$

where of course Ψ is the ground-state wave function discussed earlier. Here the $(\eta - z_i)^2$ factor implements the insertion of two flux units at η , and the factor $\exp(-\frac{1}{4}eB(|z_1|^2 + |\eta|^2 - 2z_1\eta^*))$ is the wave function (in symmetric gauge) for an electron localized near η . Thus, this wave function represents the insertion of two units of flux, plus an electron, at η . This gives a neutral excitation, which standard arguments show to be a fermion. (We will also argue below, that it carries spin $\frac{1}{2}$!) The fact that they are neutral implies that these excitations are mobile despite the magnetic background field. Therefore, although the localized version (6.1) is adequate to derive the quantum numbers, we do not expect it to represent the lowest energy configuration. In other words, these excitations, unlike the halberons, are not necessarily expected to have a narrow band structure. Accordingly a superposition of states like (6.1) but with different values of η is more appropriate – see our discussion below.

Before considering such superpositions, however, let us consider the wave function for localized pair-breaking excitations on a sphere. This will reassure us

that the pair-breakers are legitimate bulk excitations, and also allow us to draw a remarkable conclusion concerning their spin. There is a simple and natural way to put a pair of localized pair-breakers on a sphere, as follows:

$$\Psi_{\text{pinned fermions}} = \mathcal{A} (\alpha_\eta^* u_1 + \beta_\eta^* v_1)^{2N-3} (\alpha_\zeta^* u_2 + \beta_\zeta^* v_2)^{2N-3} \times \prod_{i=3}^N (\alpha_\eta v_i - \beta_\eta u_i)^2 (\alpha_\zeta v_i - \beta_\zeta u_i)^2 \Psi[(u_3, v_3), \dots, (u_N, v_N)], \quad (6.2)$$

where $(\alpha_\eta, \beta_\eta)$ and $(\alpha_\zeta, \beta_\zeta)$ are the positions of the two excitations. The power of the $(\alpha_\eta^* u_1 + \beta_\eta^* v_1)$ and $(\alpha_\zeta^* u_2 + \beta_\zeta^* v_2)$ is determined by the requirement that the total power of (u_1, v_1) (or (u_2, v_2)) is equal to the total power of $(u_i, v_i)|_{i=3, \dots, N}$. These factors represent the injection of electrons maximally localized at $(\alpha_\eta, \beta_\eta)$ and $(\alpha_\zeta, \beta_\zeta)$. It is easily verified that the power of (u_i, v_i) in $\Psi_{\text{p.b.}}$ is equal to $2N_\phi - 3$, just as in the ground state.

Thus the state of two localized neutral fermions and the ground state belong to the same Hilbert space (with fixed N and N_ϕ). The neutral fermion pair excitation is indeed an allowed excitation above the ground state. It is therefore a permissible bulk excitation. The neutrality of the quasiparticle follows from the fact that the number-flux relation is unaltered – there is neither an excess nor a deficit in electron density, induced by the existence of the pair breaking excitation.

By the way, if N were odd it would have been possible to set up a single fermion state

$$\Psi_{\text{p.f.}} = \mathcal{A} (\alpha_\eta^* u_1 + \beta_\eta^* v_1)^{2N-3} \prod_{i=2}^N (\alpha_\eta v_i - \beta_\eta u_i)^2 \Psi[(u_2, v_2), \dots, (u_N, v_N)], \quad (6.3)$$

based on the pairing wave function for $N - 1$ particles.

From the form of the wave function in (6.2), one can argue that the pair-breaking neutral fermions carry spin $\frac{1}{2}$ – that is, orbital angular momentum in the plane. Indeed, the difference between the total power of the complex conjugate of its coordinate $(\alpha_\eta^*, \beta_\eta^*)$ minus the total power of the coordinate $(\alpha_\eta, \beta_\eta)$ itself is equal to 1. That is to say, the orbital wave functions of a single neutral fermion always have half odd integer angular momentum. This means the neutral fermion pair-breaker effectively sees a unit of flux going through the sphere. Since the pair-breaker carries no electric charge, the flux must come from the coupling of the spin to the curvature of the sphere. For this coupling to generate one unit of flux, the spin of the neutral fermion must be $\frac{1}{2}$. (Note that such a spin is a quantum number of the O(2) planar rotations. It should not be confused with the angular momentum of the SU(2) spherical rotations, nor of course with an internal spin degree of freedom.)

6.3. PROPERTIES OF THE PAIR BREAKING MODES

The particular wave function in (6.2) or (6.3) may not have the lowest energy, because neutral excitations are mobile even in the presence of a magnetic field. A superposition of the states in (6.2) with different $(\alpha_\eta, \beta_\eta), (\alpha_\xi, \beta_\xi)$ will have lower energy. We will now present some conjectures as to the properties of the proper excitations. These represent little more than our present best guesses.

Since the neutral fermion carries no charge, the energy eigenstates are labeled by momentum. Thus we may take the single neutral fermion wave function at momentum k , Ψ_k , as a trial wave function and calculate the spectrum as

$$E_k = \frac{\langle \Psi_k | V | \Psi_k \rangle}{\langle \Psi_k | \Psi_k \rangle}. \tag{6.4}$$

Here Ψ_k is the Fourier transformation of the neutral fermion wave function,

$$\Psi_k[z] = \int d^2\eta \Psi_\eta[z] e^{ik\eta}, \tag{6.5}$$

and V is the interaction potential between electrons. Let us introduce

$$\tilde{V}(\eta - \xi) = \int \Psi_\eta^*[z] V \Psi_\xi[z] \prod_i d^2z_i, \tag{6.6}$$

$$g(\eta - \xi) = \int \Psi_\eta^*[z] \Psi_\xi[z] \prod_i d^2z_i.$$

Then E_k can be expressed as

$$E_k = \frac{\tilde{V}_k}{g_k}, \tag{6.7}$$

where \tilde{V}_k and g_k are the Fourier transformations of $\tilde{V}(\eta)$ and $g(\eta)$. g_k is positive since it is the norm of Ψ_k . \tilde{V}_k is also positive if the electron potential is positive definite. If our pairing state is incompressible the quasiparticle is expected to have a finite size. Therefore $\tilde{V}(\eta) \rightarrow 0$ and $g(\eta) \rightarrow 0$ as $|\eta| \rightarrow \infty$. However, in general one expects that \tilde{V} decays more slowly than g , and therefore that \tilde{V}_k is more sharply peaked at $k = 0$ than is g_k . If this is true, E_k will have a maximum at $k = 0$ and a minimum at a finite $k = k_0$. (See also the numerical evidence displayed in fig. 3b)

Our argument leading to expression (6.7) is similar to the Feynman–Bijl argument [37] for the roton spectrum in superfluid He II. Thus it is not surprising to find a similar “roton dip” in the neutral fermion spectrum. However, the “roton

dip” in our case has a new interpretation. The momentum at the minimum can approximately be interpreted as a Fermi momentum. Such an interpretation is supported by the analogy to BCS theory, which we have stressed repeatedly above. We know that in BCS theory the quasiparticle energy is minimized near the Fermi momentum.

An approximate Fermi surface is traced out by the minimum of the neutral fermion spectrum. The description as a Fermi surface will be appropriate, insofar as the energy gap can be regarded as small. This picture suggests that above the pairing temperature the specific heat of the $\nu = 1/2$ system has a linear temperature dependence.

Both our $\nu = 1/2$ pairing state and the usual superconducting state are incompressible states. Although the charged excitations in the two states are different, the low-lying neutral excitations in the two states seem to be qualitatively similar, according to this discussion.

6.4. PAIR BREAKING WITH SEPARATED QUASIHOLE

Let us consider the case where we have separated charged quasiparticles (dihalberons), which may be pinned by impurities. In this case, the pair breaking quite plausibly takes a different form. The energy of the pair breaking excitations is expected to be lower, simply because each member of the broken pair can separately take advantage of the attractive potential provided by the quasiparticles. In other words, one member may concentrate near one quasiparticle, and the other member near another quasiparticle. Indeed, in the presence of quasiholes one can suggest a much simpler broken-pair trial wave function than without them

$$\Psi_{\eta,\zeta} = \mathcal{A} \frac{1}{z_1 - \eta} \frac{1}{z_2 - \zeta} \prod_{\substack{i \text{ odd} \\ i \geq 3}} \frac{1}{z_i - z_{i+1}} \\ \times \prod_{i=1}^N (z_i - \eta)(z_i - \zeta) \prod_{i < j}^N (z_i - z_j)^2 \prod_{i=1}^N \exp\left(-\frac{eB}{4} |z_i|^2\right), \quad (6.8)$$

where η and ζ are the quasihole coordinates. It is straightforward to generalize (6.8) to the spherical geometry

$$\Psi_{\eta,\zeta} = \mathcal{A} \frac{1}{\beta_\eta u_1 - \alpha_\eta v_1} \frac{1}{\beta_\zeta u_2 - \alpha_\zeta v_2} \prod_{\substack{i \text{ odd} \\ i \geq 3}} \frac{1}{u_i v_{i+1} - u_{i+1} v_i} \\ \times \prod_{i=1}^N (\beta_\eta u_i - \alpha_\eta v_i)(\beta_\zeta u_j - \alpha_\zeta v_j) \prod_{i < j} (u_i v_j - u_j v_i)^2. \quad (6.9)$$

Mathematically, the important point is that the existence of flux tubes opens up possibilities for modifying the pfaffian factor, which is otherwise difficult. In fact we can relate the present discussion to the uniqueness argument formulated in sect. 3. There, we anticipated that modification of the pfaffian by $\phi(z) = 1/z \rightarrow 1/z + p(z)$ would correspond to pair breaking, since in the expanded product on a finite number of terms involving $p(z)$ – the broken pairs – would occur. On the sphere there is a homogeneity requirement, so that the corresponding modification has to take the form

$$\frac{1}{u_i v_j - u_j v_i} \rightarrow \frac{1}{u_i v_j - u_j v_i} + \frac{1}{\beta_\eta u_1 - \alpha_\eta v_1} \frac{1}{\beta_\zeta u_2 - \alpha_\zeta v_2} + \dots \quad (6.10)$$

But this proposed modification – the only kind consistent with homogeneity – leads to an unnormalizable wave function, when implemented around the ground state. It becomes a real possibility only when there are quasiholes at the appropriate points.

The availability of a simple and natural wave function – and the physical consideration that it ought to be energetically favorable to split up the charge distribution of a pair to track the separated quasiholes – suggests that the energy required for pair breaking may be drastically affected by the presence of quasiholes. Indeed, for the 3-body hamiltonians we introduced previously the quasiholes are zero energy states either with or without pair breaking! Of course the zero energy of the quasiholes is of limited physical means as such, because the quasiholes do not have vacuum quantum numbers (and the energy of a quasihole-quasiparticle pair is presumably *not* zero), the degeneracy between paired and pair-breaking states *is* significant.

A generalization of the considerations above, shows that when $2r$ quasiholes are present, up to r broken pairs can be accommodated naturally. One should also take into account the fact that these quasiholes can break up into halberon pairs, but we will not attempt that here.

7. Numerical experiments: model potential

In the previous chapters, we have argued for the theoretical consistency of the paired Hall state, and derived several of its qualitative properties. The techniques we used, however, do not provide a ready basis for quantitative energetic considerations. Is the paired Hall state the ground state for more realistic potentials? If so, are the low energy excitations really the peculiar ones we have argued for? These questions can be answered convincingly only by explicit, quantitative calculations. In this section and sect. 8, we will report the result of some relevant numerical

simulations *, which do seem to support many aspects of our analysis, and to suggest that the paired Hall state is a good candidate to describe experimentally attainable situations.

We have calculated the complete spectrum for small numbers of fermions on a sphere penetrated uniformly by magnetic flux, and interacting via various simple two-body potentials, using the methods of Haldane and Rezayi [15,39]. In this section we will concentrate on a class of simple model potentials, which allow us to exhibit clearly the existence of the universality class, its smooth connection to a strong pairing regime, and several of the most important qualitative features suggested by the preceding analysis. In the following chapter we shall consider more realistic potentials.

Let us consider a model in which the only interactions are pairwise interactions, and within each pair interactions occur only when they are in a relative angular momentum 1 or 3 state. The potential for angular momentum 1 is denoted V_1 and the potential for angular momentum 3 is denoted V_3 . All interactions are projected onto the first Landau level, as is appropriate if we assume that the splitting between Landau levels is much larger than the potential interaction energy.

Representative results are displayed in fig. 2. The potentials are given by $V_1 = \cos \phi$ and $V_3 = \sin \phi$, with $\phi = 0$, $\phi = 0.128\pi$, $\phi = 0.5\pi$, and $\phi = 0.628\pi$.

The most notable feature of the results is the rather clear indication of an energy gap in fig. 2b. The ground state occurs at zero angular momentum, and is therefore isotropic. Note that the relation between the amount of flux threading the sphere and the number of particles is just what is expected for the pairing state, i.e. $N_\phi = 2N - 3$. The clear gap structure departs as one deviates from this relation – and thus enters, according to our theory, sectors containing charged quasiparticles.

For other potentials, as displayed in fig. 2a, the ground state seems to occur at non-zero angular momentum. We are tempted to interpret this as the formation of a charge density wave state, or perhaps as phase separation.

The numerical experiments also confirm our prediction that the elementary flux quantum is halved due to pairing. If the flux deviates from the magic value $N_\phi = 2N - 3$ by one quantum, we predict that two identical charged elementary excitations – two halberons – should be produced. In the low-lying portion of the spectrum of fig. 3a, one notices a clear even–odd pattern for the allowed angular momenta. This is exactly what one would expect for two identical particles. (It is very familiar that for bosons the relative orbital angular momentum must be even, while for fermions it must be odd. The general result, for any statistics, is equal spacing by two units in the spectrum of allowed angular momenta.) It is also interesting that the lowest energies occur for the lowest angular momentum. This

* Related calculations have been performed by Fano et al. [38] for Coulomb interactions at $N_\phi = 2N$. No incompressible state was found.

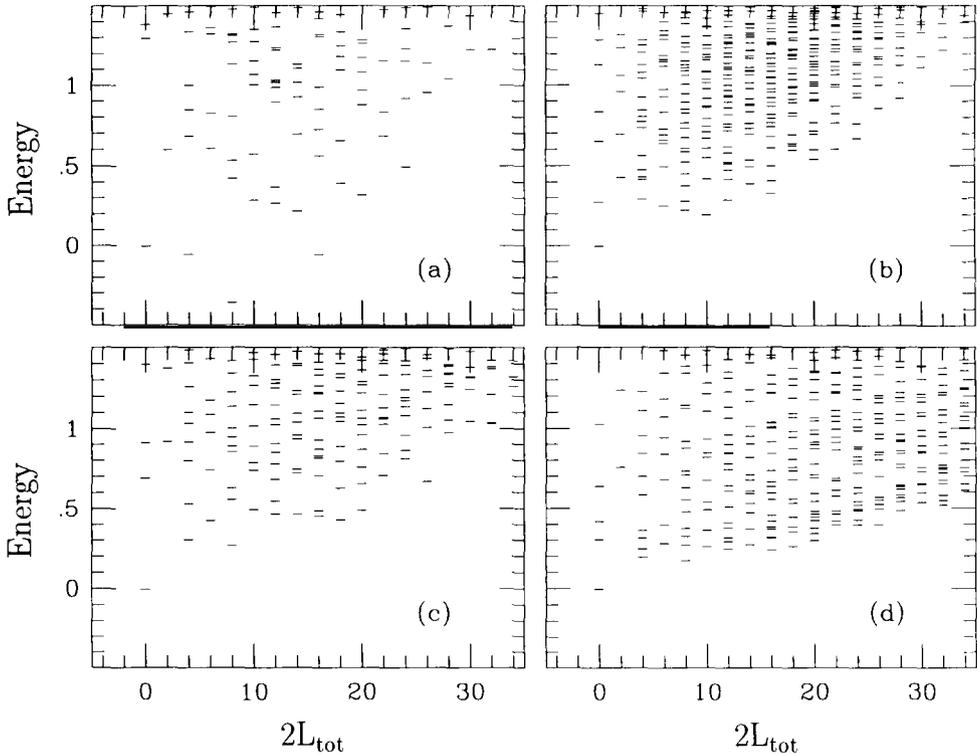


Fig. 2. The spectra of 10-electron systems on a sphere with 17 flux quanta. The interaction between electrons is $V_1 = \cos \phi$ and $V_3 = \sin \phi$ where ϕ is given by (a) $\phi = 0$, (b) $\phi = 0.128\pi$, (c) $\phi = 0.5\pi$ and (d) $\phi = 0.628\pi$.

has a simple physical meaning in the two-helberon picture. The lowest angular momentum corresponds, in real space, to two localized halberons sitting on opposite sides of the sphere. This is the configuration we expect should give the lowest energy for charged quasiparticles with repulsive interactions. On the other hand, the two quasiparticles carry maximal angular momentum when they lie on top of each other. The maximum value for a pair of halberons is thus given by $N/2$, which also agrees with our numerical result.

When we remove (or add) one electron and two units of flux from the pairing ground state, the spectrum contains a low energy branch which is separated from the “continuum” (fig. 3b). The low energy branch describes the dispersion of the neutral fermion, pair-breaking excitation. It has a minimum at finite angular momentum, as anticipated. In the weak pairing limit, such neutral excitations may have the smallest energy gap (and thus, for example, may dominate the thermal conductivity and other transport properties).

The scaling of the energy gap with N is displayed in fig. 4. As $N \rightarrow \infty$, the energy gap extrapolates to approximately 0.1 (in relative units), for $\phi = 0.128\pi$.

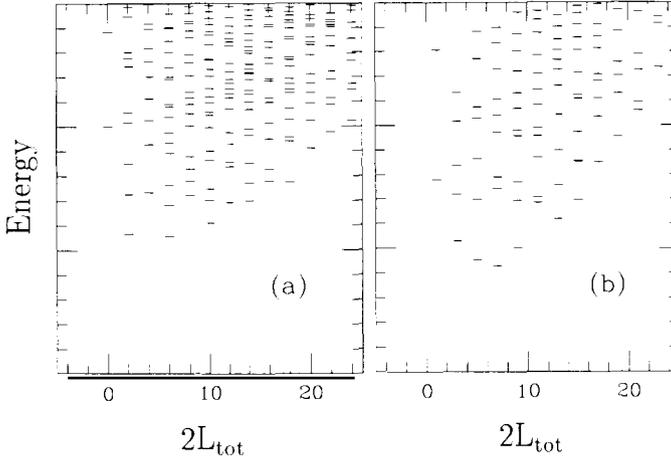


Fig. 3. The spectra of (a) 10 electrons with 18 flux quanta and (b) 9 electrons with 15 flux quanta on a sphere. The electron interaction is the same as that in fig. 2b. Note the unambiguous even-odd structure in the angular momentum dependence in (a). The neutral fermion dispersion curve can be seen clearly in (b).

This provides strong evidence for the existence of incompressible states of spinless fermions at filling fraction $1/2$, for simple repulsive potentials.

A point of considerable conceptual importance emerges upon consideration of the series of simulations on the 10-electron system depicted in figs. 2b, c and d. In this series, we have interpolated between potentials combining short-range attraction and longer-range repulsions and potentials which are purely repulsive. Throughout the interpolation, the gap never closes. This provides direct numerical evidence that the pairing $\nu = 1/2$ state for quasi-realistic interactions is in the

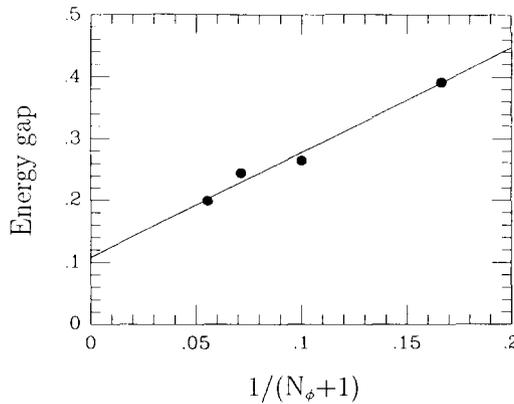


Fig. 4. The scaling of the energy gap of an electron system on sphere with $N_c = \text{even}$ and $N_\phi = 2N_c - 3$. The interaction between the electrons is the same as that in fig. 2b.

same universality class as the strong-binding effective boson Hall state proposed by Halperin. The existence of such a connection gives further, indirect but in our opinion compelling, support for the existence of the halberons with the quantum numbers of charge and the (abelian) statistics we assigned them, since these results clearly hold in the strong-pairing limit. (However we should mention that the similar calculation on the 8-electron system produced a different result. For 8 electrons, the energy gap collapses when V_1 is near zero. Numerical calculations on larger systems would be very desirable.)

8. Realistic potentials and experimental considerations

In sect. 7, we discussed numerical simulations for the pairing Hall state using a quasirealistic model potential. In this section we will discuss similar numerical simulations for realistic electron interactions, and attempt to draw some lessons relevant to practical experimental possibilities.

We have shown in ref. [10] and in sect. 7 that a strong short-range repulsion V_1 will destabilize the pairing state. For electrons in the first Landau level, it appears that for unscreened Coulomb interactions, the magnitude of V_1 is too large to favor the paired ground state (see fig. 5a). This motivates us to consider mechanisms which may serve to reduce the short range repulsive component of the Coulomb interaction.

In real samples, the short-range repulsion of the Coulomb interaction may be reduced by any of the following three effects:

- (1) Finite thickness of the electron wave function in the direction perpendicular to the plane.
- (2) The larger size of the electron wave packet in higher Landau levels.
- (3) Mixing between different Landau levels.

Here we will consider the first two effects.

To model the effect of the finite thickness of the electron wave function, we will use a screened Coulomb potential [40]

$$V(r) = \frac{e^2}{\epsilon\sqrt{r^2 + \lambda^2}} \quad (8.1)$$

and neglect finite-size corrections to the pseudopotentials [41]. In typical experiments λ/l is between 1 and 2, but for specially prepared thick samples λ/l can be as large as 4 to 5.

At filling fractions $2 < \nu < 3$, it is believed that in the strong magnetic field regime relevant to the quantized Hall effect the first Landau level is completely filled by electrons of both spins, because the gap between Landau levels is large compared to the Coulomb energy. For the same reason, the mixing between

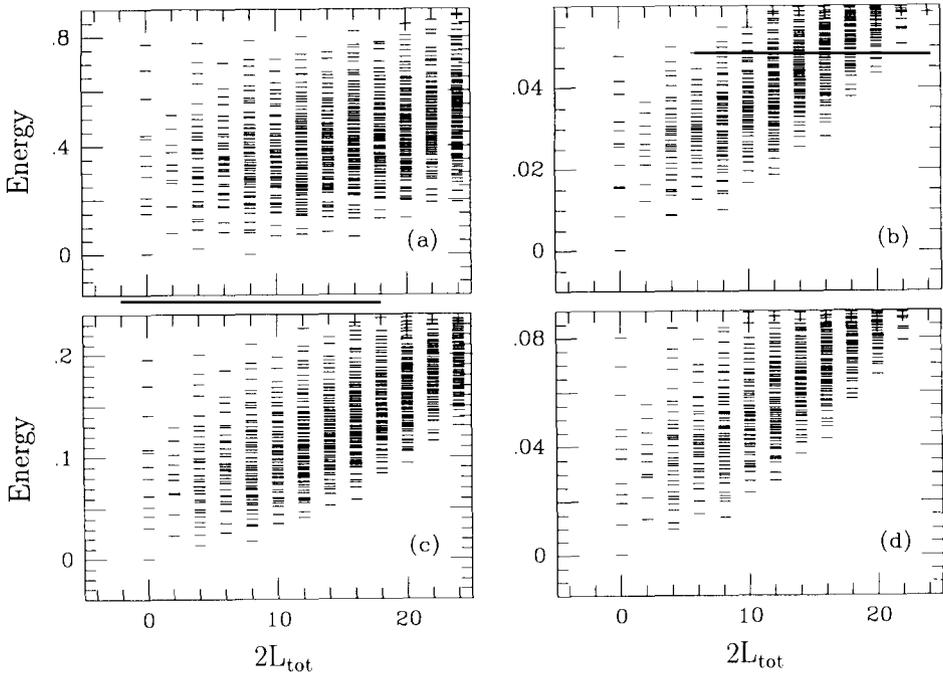


Fig. 5. The spectra of 10 electrons (a, b) in the first Landau level and (c, d) the second Landau level with 17 flux quanta. The electron interaction is given by $V = e^2/\epsilon\sqrt{r^2 + \lambda^2}$, with (a, c) $\lambda = 0$, (b) $\lambda = 4l$ and (d) $\lambda = 2l$. All energies are in (cgs) units of $e^2/\epsilon l$.

Landau levels can be ignored. Thus the system behaves effectively as if it were at filling fraction $\nu = 2$; the only significant difference is that now the electrons live in the second Landau level. This difference results in a different effective interaction between the electrons, once the projection onto wave functions within the Landau level is performed, because the wave functions in different Landau levels have different shapes. The wave functions of the second Landau level are more extended than those of the first, and therefore the projected part of short-range repulsion will be relatively less important.

Results for the energy spectrum of 10 electron on a sphere permeated with 17 flux quanta are exhibited in fig. 5. The interaction between electrons is chosen to be that in (8.1). Figs. 5a and 5b are for electrons in the first Landau level with $\lambda/l = 0$ and $\lambda/l = 4$, respectively, while figs. 5c and 5d correspond to the second Landau level with $\lambda/l = 0$ and $\lambda/l = 2$.

These finite-size calculations suggest that the pairing state may exist in real samples. For electrons in the first Landau level (corresponding to the $\nu = 1/2$ state), the pairing state may exist for especially thick samples, while for electrons in the second Landau level (corresponding to $\nu = 5/2$ state), the pairing state may quite possibly be realized in standard samples. (We will give a more detailed

discussion for the $\nu = 5/2$ state later.) We have checked the spectrums for 6-electron and 8-electron systems with flux given by $N_\phi = 2N - 3$. We find the spectra, for the corresponding interactions, to be very similar to those in fig. 5. In particular, the energy gap remains finite for the interactions with finite thickness. Because the incompressible state that we find always appears at $N_\phi = 2N - 3$, we believe it to correspond to the pairing state discussed in this paper.

We also attempted to study the scaling of the energy gaps with N . Unfortunately, the results were inconclusive. The energy gap appears to approach a value very close to zero for the cases displayed in fig. 5b and fig. 5d. Thus, based on these studies it is not clear whether or not the pairing state survives in the thermodynamic limit. Clearly, studies on larger systems would be most desirable.

The energy gap of the pairing state for the 10-electron system is found to be $\Delta_{1/2} = 0.0085 e^2/\epsilon l$ for $\lambda/l = 4$ and $\Delta_{5/2} = 0.0096 e^2/\epsilon l$ for $\lambda/l = 2$. It is interesting to compare these gaps with the $\nu = 1/3$ Laughlin state in the first and second Landau levels. For a system with 7 electrons, we obtain $\Delta_{1/3} = 0.00656 e^2/\epsilon l$ for $\lambda/l = 4$ and $\Delta_{7/3} = 0.01060 e^2/\epsilon l$ for $\lambda/l = 2$.

Now let us consider the case of unscreened Coloumb interactions ($\lambda = 0$). In the first Landau level it is clear that the $\nu = 1/2$ state is not incompressible, because of the existence of a low energy branch. The situation is less clear for the second Landau level ($\nu = 5/2$). There we can still identify a low energy branch which resembles the one in the first Landau level, but now it appears with a finite slope. This feature suggests that the $\nu = 5/2$ state for the pure Coloumb interaction is not a paired Hall state. Apparently the reduction in V_1 in the second Landau level seems thus not sufficient to stabilize a paired ground state, and finite thickness effects are required as well.

This conclusion is supported by calculations performed by Chakraborty and Pietiläinen using the torus geometry [16]. They found that the ground state of $\nu = 5/2$ system with the pure Coloumb interaction always has zero momentum on a torus, both for even and for odd numbers of electrons. If the pairing state was the ground state, the odd-number-electron system would contain a neutral fermion excitation. The ground state presumably would then exhibit non-zero momentum, as shown in fig. 3b.

We also have argued that the paired Hall state has eight-fold ground state degeneracy on a torus in the thermodynamic limit. For a finite system the degeneracy is split by an amount of order $e^{-L/\xi}$ where ξ is the typical length scale of pairing state [30]. Numerical calculations for the unscreened Coulomb interaction have not exhibited these eight closely degenerate ground states.

The results presented above suggest the following possible scenario for the $\nu = 5/2$ electron system. Above a certain critical thickness the ground state of the system is the pairing state, and demonstrates the quantum Hall effect. Below the critical thickness the pairing state is unstable and the system may be in a compressible state. Certainly this scenario is consistent with our numerical results,

but we cannot claim that our simulations, based on such small systems, establish it definitively.

In the discussion so far we have ignored the spin degree of freedom of the electrons. Haldane and Rezayi [15] have suggested a spin singlet state to describe the observed incompressible state at $\nu = 5/2$. However, the possibility of a spin singlet state has not received sufficient support from numerical calculations for quasi-realistic potentials. For the pure Coloumb interaction in the second Landau level, it has been shown that the ground state is fully spin polarized, even if the Zeeman splitting is not taken into account. Finite thickness calculations with reversed spins are needed to reach a more conclusive result. It presently appears to us that, taking the known numerical results at face value, the paired Hall state is the best bet to describe the incompressible $\nu = 5/2$ state.

It is noteworthy that the spin singlet state and the pairing state have very different behavior in the high magnetic field limit. The spin singlet state will eventually be destroyed by the high magnetic field, while the pairing state will become more stable, because of the increase of λ/l . Experiments in the high field limit for high mobility samples could distinguish between these alternatives.

The energy gap of the $\nu = 5/2$ state has been measured to be $\Delta_{\text{exp}} \sim 200$ mK (for a magnetic field $B \sim 5$ T). The gap for the 10-electron system is $\Delta_{5/2} \sim 1$ K (we have taken $\epsilon = 10$). $\Delta_{5/2}$ can be regarded as an upper bound of the gap of the pairing state. We see that the value of the $\Delta_{5/2}$ is reasonably consistent with the experimental value.

Finally, let us briefly comment on the tilted magnetic field experiments performed by Eisenstein and coworkers [18]. In these experiments, the magnetic field has a component tangent to the plane of the 2d electron gas, in addition to the perpendicular component. The magnetic field component tangent to the plane tends to have the same effect as reducing the thickness, since the particles do not like to cross field lines. Because the thickness of the real samples is of the same order as the magnetic length, the effective thickness is sensitive to the tilt angle. According to recent numerical work [42], the effective thickness may be reduced by a factor 2 or 3 if the sample is tilted by, say, 60° . Therefore the paired Hall state and the associated Hall plateau may be destroyed by tilting the magnetic field which reduce the thickness.

In fact, the observed incompressible $\nu = 5/2$ state does seem to be disrupted by a tilted magnetic field. The conventional interpretation of this observation has been that it provides evidence that the ground state is not spin polarized. For if it were spin polarized, it might seem that the tilted field would be accomodated simply by appropriate re-alignment of the spin direction, without loss of correlation energy. However of course the Zeeman coupling is not the only coupling of the electrons to the magnetic field, as we have just remarked, so the conventional interpretation is not entirely well grounded. Indeed the picture described above, based on our numerical simulations, is very similar to what has been observed in

the experiments at filling fraction $5/2$. Thus we are tempted to conclude that the tilted-field experiments do not necessarily imply, as is generally asserted, that the electrons in the second Landau level are spin unpolarized.

In sum, existing experiments are not manifestly inconsistent with the assumption that the electrons in the second Landau level form a spin polarized paired Hall state at filling fraction $\nu = 5/2$. In fact, existing numerical calculations seem to point toward such a possibility. Experiments at higher magnetic fields could help settle the issue. The spin-polarized pairing state would tend to be more favorable in high magnetic fields, due to an increase in the thickness/magnetic length ratio, while a spin-singlet pairing state would tend to be destabilized, simply due to the Zeeman effect.

References

- [1] K. v. Klitzing, G. Dorda and M. Pepper, Phys. Rev. Lett. 45 (1980) 494; D.C. Tsui, H.L. Störmer and A.C. Gossard, Phys. Rev. Lett. 48 (1982) 1559
- [2] R.E. Prange and S.M. Girvin, ed., The quantum Hall effect (Springer, New York, 1987); T. Chakraborty and P. Pietiläinen, The fractional quantum Hall effect (Springer, New York, 1988)
- [3] R.B. Laughlin, Phys. Rev. B23 (1981) 5632; B.I. Halperin, Phys. Rev. B25 (1982) 2185
- [4] R.B. Laughlin, Phys. Rev. Lett. 50 (1983) 1395; Surf. Sci. 142 (1984) 163
- [5] B.I. Halperin, Helv. Phys. Acta 56 (1983) 75
- [6] F.D.M. Haldane, Phys. Rev. Lett. 51 (1983) 605
- [7] B.I. Halperin, Phys. Rev. Lett. 52 (1984) 1583
- [8] M. Greiter and F. Wilczek, Mod. Phys. Lett. B4 (1990) 1063
- [9] M. Greiter and F. Wilczek, Nucl. Phys. B370 (1992) 577
- [10] M. Greiter, X.-G. Wen and F. Wilczek, Phys. Rev. Lett. 66 (1991) 3205
- [11] G. Canright and S. Girvin, Int. J. Mod. Phys. B3 (1989) 1943, see especially footnote 30
- [12] G. Moore and N. Read, Nucl. Phys. B360 (1991) 362
- [13] R. Willett, J.P. Eisenstein, H.L. Störmer, D.C. Tsui, A.C. Gossard and J.H. English, Phys. Rev. Lett. 59 (1987) 1776; R.G. Clark, R.J. Nicholas, A. Usher, C.T. Foxon and J.J. Harris, Surf. Sci. 170 (1986) 141
- [14] H.W. Jiang, H.L. Störmer, D.C. Tsui, L.N. Pfeiffer and K.W. West, Phys. Rev. B40 (1989) 12013; R.L. Willett, M.A. Paalanen, R.R. Ruel, K.W. West, L.W. Pfeiffer and D.J. Bishop, Phys. Rev. Lett. 65 (1990) 112
- [15] F.D.M. Haldane and E.H. Rezayi, Phys. Rev. Lett. 60 (1988) 956
- [16] T. Chakraborty and P. Pietiläinen, Phys. Rev. B38 (1988) 10097
- [17] A.H. MacDonald, D. Yoshioka and S.M. Girvin, Phys. Rev. B39 (1989) 8044
- [18] J.P. Eisenstein, R. Willett, H.L. Störmer, D.C. Tsui, A.C. Gossard and J.H. English, Phys. Rev. Lett. 61 (1988) 997
- [19] R. Tao and Y.-S. Wu, Phys. Rev. B31 (1985) 1191
- [20] J. Leinaas and J. Myrheim, Il Nuovo Cimento 37B (1977) 1; G. Goldin, R. Menikoff and D. Sharp, J. Math. Phys. 21 (1981) 650
- [21] F. Wilczek, Phys. Rev. Lett. 48 (1982) 1144; 49 (1982) 957
- [22] D.P. Arovas, R. Schrieffer, F. Wilczek and A. Zee, Nucl. Phys. B251 (1985) 117
- [23] A.L. Fetter, C.B. Hanna and R.B. Laughlin, Phys. Rev. B39 (1989) 9679
- [24] Y.-H. Chen, F. Wilczek, E. Witten and B.I. Halperin, Int. J. Mod. Phys. B3 (1989) 1001
- [25] L.N. Cooper, Phys. Rev. 104 (1956) 1189; J. Bardeen, L.N. Cooper and J.R. Schrieffer, Phys. Rev. 106 (1957) 162; 108 (1957) 1175

- [26] F. Dyson, quoted in J.R. Schrieffer, *Theory of superconductivity* (Benjamin-Cummings, Reading, MA, 1983) p. 42
- [27] S. Girvin, A. MacDonald, M.P.A. Fisher, S.-J. Rey and J. Sethna, *Phys. Rev. Lett.* 65 (1990) 1671
- [28] B.M. McCoy and T.T. Wu, *The two-dimensional Ising model* (Harvard, Cambridge, MA, 1973)
- [29] S.A. Trugman and S. Kivelson, *Phys. Rev.* B31 (1985) 5280
- [30] X.G. Wen and Q. Niu, *Phys. Rev.* B41 (1990) 9377
- [31] F.D.M. Haldane and E.H. Rezayi, *Phys. Rev.* B31 (1985) 2529
- [32] D. Mumford, *Tata lectures on theta*, Vols. I and II (Birkhäuser, Basel, 1983)
- [33] G. Jones and D. Singerman, *Complex functions* (Cambridge University Press, Cambridge, 1987)
- [34] C.N. Byers and C.N. Yang, *Phys. Rev. Lett.* 7 (1961) 46
- [35] T. Einarsson, *Phys. Rev. Lett.* 64 (1990) 1995
- [36] X.G. Wen, E. Dagotto and E. Fradkin, *Phys. Rev.* B42 (1990) 6110;
T.D. Imbo and J. March-Russell, *Phys. Lett.* B252 (1990) 84
- [37] R.P. Feynman, in *Progress in low temperature physics*, ed. C.J. Gorter (North Holland, Amsterdam, 1955)
- [38] G. Fano, F. Ortolani and E. Tosatti, *Nuovo Cimento* 9D (1987) 1337
- [39] F.D.M. Haldane and E.H. Rezayi, *Phys. Rev. Lett.* 54 (1985) 237
- [40] F.C. Zhang and S. Das Sarma, *Phys. Rev.* B33 (1986) 2903
- [41] S. He, F.C. Zhang, X.C. Xie and S. Das Sarma, *Phys. Rev.* B42 (1990) 11376
- [42] T. Chakraborty and P. Pietiläinen, *Phys. Rev.* B39 (1989) 7971