

## Non-Abelian Statistics in a Quantum Antiferromagnet

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We propose a novel spin liquid state for a spin  $S = 1$  antiferromagnet in two dimensions. The ground state violates  $P$  and  $T$ , is a spin-singlet, and is fully invariant under the lattice symmetries. The spinon and holon excitations are deconfined and obey non-Abelian statistics. We present preliminary numerical evidence that the universality class of this topological liquid can be stabilized by a local Hamiltonian involving three-spin interactions.

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*Introduction.*—Fractional quantization in two-dimensional quantum liquids is witnessing a renaissance of interest in present times. The field started about a quarter of a century ago with the discovery of the fractional quantum Hall effect, which was explained by Laughlin [1] in terms of an incompressible quantum liquid supporting fractionally charged (vortex or) quasiparticle excitations. When formulating a hierarchy of quantized Hall states [2,3] to explain the observation of quantized Hall states at other filling fractions, Halperin [3] noted that these excitations obey fractional statistics [4], and are hence conceptually similar to the charge-flux tube composites introduced by Wilczek two years earlier [5].

The interest was renewed a few years later, when Anderson [6] proposed that hole-doped Mott insulators, and, in particular, those described by the  $t$ - $J$  model universally believed to describe the CuO planes in high  $T_c$  superconductors, can be described in terms of a spin liquid (i.e., a state with strong, local antiferromagnetic correlations but without long range order), which would likewise support fractionally quantized excitations. In this proposal, the excitations are spinons and holons, which carry spin  $1/2$  and no charge or no spin and charge  $+e$ , respectively. The fractional quantum number of the spinon is the spin, which is half integer while the Hilbert space (for the undoped system) is built up of spin flips, which carry spin one. One of the earliest proposals for a spin liquid supporting deconfined spinon and holon excitations is the (Abelian) chiral spin liquid (CSL). Following up on an idea by D. H. Lee, Kalmeyer and Laughlin [7] proposed that a quantized Hall wave function for bosons could be used to describe the amplitudes for spin-flips on a lattice. The CSL state did not turn out to be relevant to CuO superconductivity, but remains one of very few examples of two-dimensional spin liquids with fractional quantization. The other established examples are the resonating valence bond (RVB) phases of the Rokhsar-Kivelson model [8] on the triangular lattice identified by Moessner and Sondhi [9] and of the Kitaev model [10].

The present renaissance of interest in fractional quantization is due to possible applications of states supporting excitations with *non-Abelian statistics* to the rapidly evol-

ving field of quantum computation and cryptography. The paradigm for this universality class is the Pfaffian state introduced by Moore and Read [11] in 1991. The state was proposed to be realized at the experimentally observed fraction  $\nu = 5/2$  [12] (i.e., at  $\nu = 1/2$  in the second Landau level) by Wen, Wilczek, and one of us [13], a proposal which recently received experimental support through the direct measurement of the quasiparticle charge [14,15]. Pfaffian type states are further conjectured to be realized for one-dimensional bosons with three-body hard core interactions in general [16]. The Moore-Read state possesses  $p + ip$  wave pairing correlations. The flux quantum of the vortices is one half of the Dirac quantum, which implies a quasiparticle charge of  $e/4$ . Like the vortices in a  $p$  wave superfluid, these quasiparticles possess Majorana-fermion states [17] at zero energy (i.e., one fermion state per pair of vortices, which can be occupied or unoccupied). A Pfaffian state with  $2L$  spatially separated quasiparticle excitations is hence  $2^L$  fold degenerate, in accordance with the dimension of the internal space spanned by the zero energy states. While adiabatic interchanges of quasiparticles yield only overall phases in Abelian quantized Hall states, braiding of half vortices of the Pfaffian state will in general yield nontrivial changes in the occupations of the zero energy states [18,19], which render the interchanges noncommutative or non-Abelian. In particular, the internal state vector is insensitive to local perturbations—it can *only* be manipulated through braiding of the vortices. These properties together render non-Abelions preeminently suited for applications as protected qubits in quantum computation [20]. Non-Abelian anyons further appear in certain other quantum Hall states including the Read-Rezayi states [21], in the Kitaev model [10], and in the Yao-Kivelson model [22].

In this Letter, we propose a novel chiral spin liquid state for an  $S = 1$  antiferromagnet. The spinon and holon excitations of this state are deconfined and obey non-Abelian statistics, with the braiding governed by Majorana-fermion states. The state violates time reversal ( $T$ ) and parity ( $P$ ), is a spin singlet, can be formulated on any lattice type, and fully respects all the lattice symmetries. The state possesses a threefold topological degeneracy on the torus

geometry. We provide preliminary numerical evidence that the state can be stabilized on the triangular lattice by a local Hamiltonian involving three-spin interactions. Finally, we consider the possibility that spinons in spin liquids with spin larger than 1/2 might obey non-Abelian statistics in general.

*Non-Abelian chiral spin liquid state.*—The state we propose is most easily written down for a circular droplet with open boundary conditions occupying  $N$  sites of a triangular or square lattice  $S = 1$  antiferromagnet. The wave function for renormalized spin flips,

$$\psi_0[z_i] = \text{Pf}\left(\frac{1}{z_j - z_k}\right) \prod_{i<j}^N (z_i - z_j) \prod_{i=1}^N G(z_i) e^{-(\pi/2)|z_i|^2} \quad (1)$$

is given by a bosonic Pfaffian state in the complex coordinates  $z \equiv x + iy$  supplemented by a gauge phase  $G(\eta_\alpha)$ . The Pfaffian is given by the fully antisymmetrized sum over all possible pairings of the  $N$  coordinates,

$$\text{Pf}\left(\frac{1}{z_i - z_j}\right) \equiv \mathcal{A}\left\{\frac{1}{z_1 - z_2} \dots \frac{1}{z_{N-1} - z_N}\right\}. \quad (2)$$

The “particles”  $z_i$  represent renormalized spin flips acting on a vacuum with all spins in the  $S^z = -1$  state,

$$|\psi_0\rangle = \sum_{\{z_1, \dots, z_N\}} \psi_0(z_1, \dots, z_N) \tilde{S}_{z_1}^+ \dots \tilde{S}_{z_N}^+ |-1\rangle_N, \quad (3)$$

where the sum extends over all possibilities of distributing the  $N$  particles over the  $N$  lattice sites allowing for double occupation, and

$$\tilde{S}_\alpha^+ \equiv \frac{S_\alpha^z + 1}{2} S_\alpha^+, \quad |-1\rangle_N \equiv \otimes_{\alpha=1}^N |1, -1\rangle_\alpha. \quad (4)$$

The lattice may be anisotropic; we have chosen the lattice constants such that the area of the unit cell spanned by the primitive lattice vectors is set to unity. For a triangular or square lattice with lattice positions given by  $\eta_{n,m} = na + mb$ , where  $a$  and  $b$  are the primitive lattice vectors in the complex plane,  $G(\eta_{n,m}) = (-1)^{(n+1)(m+1)}$  [7,23].

*Singlet property.*—While the topological properties, and, in particular, the non-Abelian statistics of the fractionalized excitations of (1), are suggestive to those familiar with Pfaffian states, the invariance under spin rotation and lattice symmetries is less so. We content ourselves here with a direct proof of the singlet property, which at the same time serves to motivate the necessity for the renormalization of the spin-flip operators (4).

Since  $S_{\text{tot}}^z |\psi_0\rangle = 0$  by construction, it is sufficient to show  $S_{\text{tot}}^- |\psi_0\rangle = 0$ . Note first that when we substitute (1) with (2) into (3), we may omit the antisymmetrization  $\mathcal{A}$  in (2), as it is taken care by the commutativity of the bosonic operators  $\tilde{S}_\alpha^+$ . (Throughout this Letter, we do not keep track of overall normalization factors.) Let  $\tilde{\psi}_0$  be  $\psi_0$  without the operator  $\mathcal{A}$  in (2). Since  $\tilde{\psi}_0[z_i]$  is still symmetric under interchange of pairs, we may assume that a

spin-flip operator  $S_\alpha^-$  acting on  $|\tilde{\psi}_0\rangle$  will act on the pair  $(z_1, z_2)$ :

$$\begin{aligned} S_\alpha^- |\tilde{\psi}_0\rangle &= \sum_{\{z_3, \dots, z_N\}} \left\{ \sum_{z_2 (\neq \eta_\alpha)} \tilde{\psi}_0(\eta_\alpha, z_2, z_3, \dots) S_\alpha^- \tilde{S}_\alpha^+ \tilde{S}_{z_2}^+ \right. \\ &\quad + \sum_{z_1 (\neq \eta_\alpha)} \tilde{\psi}_0(z_1, \eta_\alpha, z_3, \dots) S_\alpha^- \tilde{S}_{z_1}^+ \tilde{S}_\alpha^+ \\ &\quad \left. + \tilde{\psi}_0(\eta_\alpha, \eta_\alpha, z_3, \dots) S_\alpha^- (\tilde{S}_\alpha^+)^2 \right\} \tilde{S}_{z_3}^+ \dots |-1\rangle_N \\ &= \sum_{\{z_3, \dots, z_N\}} \left\{ \sum_{z_2} 2 \tilde{\psi}_0(\eta_\alpha, z_2, z_3, \dots) \tilde{S}_{z_2}^+ \right\} \tilde{S}_{z_3}^+ \dots |-1\rangle_N, \end{aligned}$$

where we have used

$$S_\alpha^- (\tilde{S}_\alpha^+)^n |1, -1\rangle_\alpha = n (\tilde{S}_\alpha^+)^{n-1} |1, -1\rangle_\alpha.$$

This implies  $S_{\text{tot}}^- |\psi_0\rangle = \sum_{\alpha=1}^N S_\alpha^- |\psi_0\rangle = 0$  if and only if  $\sum_{\alpha=1}^N \tilde{\psi}(\eta_\alpha, z_2, z_3, \dots) = 0 \forall z_2, z_3, \dots, z_N$ . The Perelomov identity [24] states that this holds for infinite lattice sums of  $e^{-(\pi/2)|\eta_\alpha|^2} G(\eta_\alpha)$  times any analytic function of  $\eta_\alpha$ . Since the correlations in (1) are entirely local, the singlet property of the infinite system implies that the state is locally a singlet as well.

*Generation from filled Landau levels.*—Rather than proceeding in verifying invariance properties of the non-Abelian CSL state (1) directly, we motivate them indirectly through demonstrating that the state can alternatively be generated through successive projection via the abelian CSL from the wave functions of a filled lowest Landau level (LLL). If we choose an auxiliary magnetic field with a strength of one half of a Dirac flux quanta per lattice site, the wave function for a circular droplet of  $M = \frac{N}{2}$  fermions filling the LLL is given by

$$\phi[z_i] = \prod_{i<j}^M (z_i - z_j) \prod_{i=1}^M e^{-(\pi/4)|z_i|^2}. \quad (5)$$

The (Abelian) CSL state for spin  $S = \frac{1}{2}$  [7], which was recently shown to be the unique and exact ground state of a local Hamiltonian [25],

$$\psi_0^{\text{CSL}}[z_i] = \prod_{i<j}^M (z_i - z_j)^2 \prod_{i=1}^M G(z_i) e^{-(\pi/2)|z_i|^2}, \quad (6)$$

where the particles  $z_i$  describe spin flips  $S_\alpha^+$  acting on a “vacuum” state with all the spins  $\downarrow$ , and  $G(\eta_\alpha)$  is as above, can be generated by Gutzwiller projection of the LLL (5) filled once with  $\uparrow$  and once with  $\downarrow$  spin fermions [26,27]:

$$|\psi_0^{\text{CSL}}\rangle = \sum_{\{z,w\}} \phi[z_i] \phi[w_j] c_{z_1 \uparrow}^\dagger \dots c_{z_M \uparrow}^\dagger c_{w_1 \downarrow}^\dagger \dots c_{w_M \downarrow}^\dagger |0\rangle, \quad (7)$$

where the sum extends over all partitions of the lattice sites into  $z$ 's and  $w$ 's and the  $c^\dagger$ 's are fermion creation operators. We can rewrite the CSL state vector in terms of Schwinger bosons  $a^\dagger$  and  $b^\dagger$ ,

$$|\psi_0^{\text{CSL}}\rangle = \Psi^{\text{CSL}}[c_\uparrow^\dagger, c_\downarrow^\dagger]|0\rangle = \Psi^{\text{CSL}}[a^\dagger, b^\dagger]|0\rangle,$$

provided we define  $\Psi_0^{\text{CSL}}[c_\uparrow^\dagger, c_\downarrow^\dagger]$  such that the operators are ordered according to a fixed labeling of the lattice sites. The non-Abelian CSL state (1) can thus alternatively be written as a symmetrization over two Abelian CSL states

$$|\psi_0\rangle = (\Psi^{\text{CSL}}[a^\dagger, b^\dagger])^2|0\rangle. \quad (8)$$

To verify (8), use  $\frac{1}{\sqrt{2}}(a^\dagger)^n(b^\dagger)^{(2-n)}|0\rangle = (\tilde{S}^+)^n|0\rangle$  and

$$\begin{aligned} \mathcal{S} \prod_{i<j,1}^M (z_i - z_j)^2 \prod_{i<j,M+1}^{2M} (z_i - z_j)^2 \\ = \text{Pf} \left( \frac{1}{z_i - z_j} \right) \prod_{i<j}^{2M} (z_i - z_j), \end{aligned}$$

where  $\mathcal{S}$  indicates symmetrization. Since the LLL states (5) are (on compact surfaces) translationally and rotationally invariant modulo gauge transformations in the auxiliary magnetic field, and (7) is manifestly gauge covariant, both the CSL states (1) and (6) are invariant under lattice transformations. Note that this projective construction also implies the singlet property of the CSL states. It can be used to formulate the CSL states on any lattice, and to generalize them to arbitrary spin:

$$|\psi_0^{\text{Spin } S}\rangle = (\Psi^{\text{CSL}}[a^\dagger, b^\dagger])^{2S}|0\rangle. \quad (9)$$

Written in terms of (then differently) renormalized spin flip particles, the wave function generalizes from a bosonic Pfaffian state for  $S = 1$  to bosonic Read-Rezayi states [21] for  $S > 1$ .

*Non-Abelian spinon and holon excitations.*—The spinon excitations of (1) are analogous to the half vortex quasiparticles of the Moore-Read quantum Hall state [11]. For example, to create 4  $\downarrow$  spin spinons at locations  $\eta_1, \eta_2, \eta_3$ , and  $\eta_4$ , we simply insert half quantum vortices inside the Pfaffian (2), which then becomes

$$\text{Pf} \left( \frac{(z_i - \eta_1)(z_j - \eta_2)(z_i - \eta_3)(z_j - \eta_4) + (i \leftrightarrow j)}{z_i - z_j} \right). \quad (10)$$

The braiding properties of the spinons are insensitive to the spinon spin, and are exactly those of the Moore-Read quasiparticles [17–19]. The proof of the singlet property given above can be extended to show that a pair of  $\downarrow$  spin spinons transforms as an  $S = 1$  triplet excitation, which implies that each spinon carries spin  $S = \frac{1}{2}$ . With the implicit assumption that the  $S = 1$  spins on each lattice site consist of two electrons in triplet configurations, we can create holon excitations by annihilating  $\downarrow$  spin electrons on sites with  $\downarrow$  spin spinons. The braiding properties of the holons are equivalent to those of the spinons.

*Model Hamiltonian.*—The first question with regard to possible applications of our state to quantum computation is whether a state belonging to the universality class de-

scribed by (1) can be stabilized through a local Hamiltonian. While we are short of a definite answer, we have done our best to address the question numerically. To begin with, we have written out the state (1) for an isotropic, triangular lattice with 16 sites and periodic boundary conditions, which imply a threefold topological degeneracy [13]. We then numerically optimized the coefficients of a set of local spin interaction terms (see Fig. 1) such that the ground state of our trial Hamiltonian is energetically closest to a suitable linear combination of the three (in the thermodynamic limit degenerate) Pfaffian states, which we then compare to the exact eigenstates. As shown in Fig. 2, the three lowest energy eigenstates of our trial Hamiltonian have a significant overlap with the Pfaffian states (i.e., 0.959, 0.964, and 0.934 in a fully symmetry reduced  $S_{\text{tot}}^z = 0$  Hilbert space with dimension 163 101), which suggests that the exact states belong to the same universality class. Note that the coefficients in Fig. 1 fall off rapidly with the distance. Small variation of the parameters induce no sensitive change in the overlaps, which indicates that the non-Abelian CSL state is stabilized throughout a finite region in parameter space. Our evidence is unfortunately not conclusive as the three CSL states are not separated by a large gap from the remainder of the spectrum, which indicates that the system we can access numerically is too small to settle the question unambiguously.

*Experimental realization.*—Recent work on polar molecules in optical lattices [28], but, in particular, on engineering three-body interactions [29], suggests that a realization of the non-Abelian CSL proposed here might be possible at some stage in the future.

*Non-Abelian spinons in general.*—Efforts to understand high  $T_c$  superconductivity in terms of an RVB spin liquid have revealed a general connection between  $d$ -wave superconductors and  $S = \frac{1}{2}$  spin liquids on the square lattice [30,31]. In particular, a wide class of (undoped)  $S = \frac{1}{2}$  spin liquids can be obtained by Gutzwiller projection from the wave function of a  $d$ -wave superconductor with

operator	configurations	coefficients
$S_i S_j$		1,931 0,079
$S_i(S_j \times S_k)$		0,970 0,344
$(S_i S_j)^2$		-0,513 -0,241 -0,086
$(S_i S_j)(S_i S_k)$		-0,137 -0,023 -0,089 -0,017

FIG. 1 (color online). The 11 interaction terms included in our trial Hamiltonian with the numerically optimized coefficients (see text).

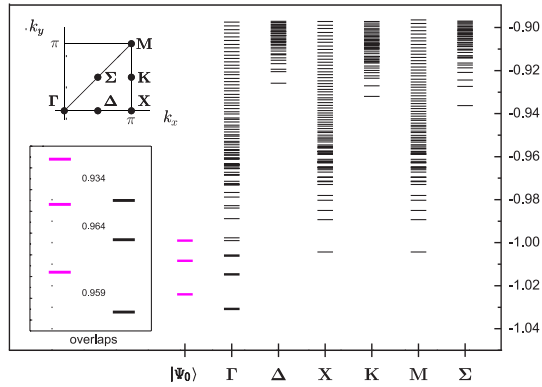


FIG. 2 (color online). Spectral plot of our trial Hamiltonian in comparison with the energy expectations values for the three (in the infinite system topologically degenerate) Pfaffian ground states at the  $\Gamma$  point. The inset shows the overlap of the Pfaffian states with the three lowest states of our Hamiltonian.

suitably chosen parameters. This suggests a general connection between the (Abelian) vortices of the superconductor and the (Abelian) spinons in the spin liquid. If one Gutzwiller projects a  $d + id$  wave superconductor with suitably chosen parameter on a square lattice, one obtains exactly the CSL state (6).

The  $p + ip$  pairing correlations in the non-Abelian CSL state (1) introduced above suggest a similar correspondence between the non-Abelian vortices of the superconductor and the non-Abelian spinon excitations (10). As in the Abelian case  $S = \frac{1}{2}$ , the  $P$  and  $T$  violation of the state appears to be necessary for the spinon to be deconfined, but does not seem essential to the topological properties. It is possible that there is a general connection between  $p$ -wave superfluids and  $S = 1$  spin liquids, in that the non-Abelian braiding properties of the vortices of the superfluid are also general properties of the spinons in  $S = 1$  antiferromagnets. True, the spinons will only be free under special circumstances, and the propensity to be confined will only increase with the spin  $S$ . Even in an ordered antiferromagnet, however, spinons (and holons) are the fields appropriate for describing the physics at sufficiently high energy scales, i.e., energies above the ordering temperature.

We will show elsewhere that the total dimension of the Hilbert space spanned by the ground state plus all states with different numbers of spinons for the spin liquid we propose is  $3^N$ , as required for an  $S = 1$  system with  $N$  sites. It is hence possible that the non-Abelian statistics for the spinons is not only a sufficient, but even necessary condition for the state counting to work out consistently. (Haldane [32] has shown that the state counting for  $S = \frac{1}{2}$  spin liquids works out consistently if one assumes Abelian half-Fermi statistics for the spinons.)

*Conclusion.*—In this work, we have constructed an  $S = 1$  CSL and argued that its spinon and holon excitations obey non-Abelian statistics. We have used exact diagonal-

ization studies to obtain preliminary indication that the state can be stabilized on a  $S = 1$  triangular lattice.

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