Statistical phases and momentum spacings for one-dimensional anyons

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Anyons and fractional statistics are by now well established in two-dimensional systems. In one dimension, fractional statistics has been established so far only through Haldane's fractional exclusion principle, but not via a fractional phase the wave function acquires as particles are interchanged. At first sight, the topology of the configuration space appears to preclude such phases in one dimension. Here we argue that the crossings of one-dimensional anyons are always unidirectional, which makes it possible to assign phases consistently and hence to introduce a statistical angle θ. The fractional statistics then manifests itself in fractional spacings of the single-particle momenta of the anyons when periodic boundary conditions are imposed. These spacings are given by $\Delta p=2\pi\hbar/L(|\theta|/\pi+\text{non-negative integer})$ for a system of length L. This condition is the analog of the quantization of relative angular momenta according to $l_z=\hbar(-\theta/\pi+2\times\text{integer})$ for two-dimensional anyons.

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I. INTRODUCTION

The concept of fractional statistics,[1,2] as introduced by Leinaas and Myrheim[3] (see also Goldin *et al.*[4]) and Wilczek[5] has generically been associated with identical particles in *two space dimensions*. It is intimately related to the topology of the configuration space or the existence of fractional relative angular momentum. Angular momentum does not exist in one dimension and is quantized in units of $\hbar/2$ in three dimensions due to the commutation relations of the three generators of rotations. In two dimensions, however, there is only one generator, L_z, which may have arbitrary eigenvalues l_z. Wilczek[5] proposed that two-dimensional (2D) anyons with statistical parameter θ and relative angular momenta $l_z=\hbar(-\theta/\pi+2\times\text{integer})$ may be realized by particle flux-tube composites, attaching magnetic flux $\Phi=2\theta\hbar c/e=\theta/\pi\Phi_0$ to bosons of charge e. The choices $\theta=0$ and $\theta=\pi$ correspond to bosons and fermions, respectively.

More fundamentally, the possibility of fractional statistics arises in 2D because one can associate a winding number with paths interchanging particles. The sum over paths in the many-particle path integral consists of infinitely many topologically distinct sectors, which correspond to the different winding configurations of the particles around each other. By the rules of quantum mechanics, one is allowed to assign different weights to distinct sectors, provided these weights satisfy the composition principle. In particular, one may assign a phase factor $e^{\pm i\theta}$ for each (counter)clockwise interchange of two particles. This choice corresponds to Abelian anyons with statistical parameter θ if the bare particles are bosons. The implicit assumption that the world lines never cross holds automatically for all values $\theta\neq 0 \bmod 2\pi$ due to the nonvanishing relative angular momentum alluded to above. In three or higher dimensions, the only topologically inequivalent sectors correspond to interchanges of particles, and the only consistent choices for the statistics are bosons and fermions. In one-dimensional (1D), the situation is alike if particles are allowed to pass through each other and trivial if they are not. In either case, the topology appears to preclude the possibility of one-dimensional anyons.

The association of anyons with 2D, however, was challenged in 1991 by Haldane,[6] who generalized the notion of fractional statistics to arbitrary dimensions by defining statistics through a fractional and hence generalized Pauli exclusion principle. According to his definition, the statistics of anyons is given by a rational "exclusion" parameter $g=p/q$ (with p and q integers) which states that the creation of q anyons reduces the number of single-particle states additional anyons could be placed in by p. In particular, Haldane *et al.*[7–10] showed that the spinons in the Haldane-Shastry model (HSM), a spin-1/2 chain with Heisenberg interactions which fall off as $1/r^2$ with the distance, obey the half-Fermi exclusion statistics. Haldane observed that for a chain with N sites, the number of single-particle states available to additional spinons is given by $M+1$, where M is the number of up or down spins in the uniform singlet liquid, which in the presence of N_{sp} spinons is given by $M=(N-N_{sp})/2$. The creation of two spinons hence reduces the number of available states by 1, which implies $g=1/2$. (Note that since there are always fewer single-spinon states as there are sites, localized spinon states cannot be orthogonal.) Haldane[7] further demonstrated that the dimension of the Hilbert space spanned by the ground state and all possible many-spinon eigenstates of the model is 2^N, as required for a spin-1/2 system with N sites. The concept of fractional statistics hence was established in a one-dimensional system, but it appeared that it could only be defined through an exclusion principle. Moreover, Haldane[6] observed that the two definitions of statistics do not always match, as hard-core bosons in 2D with magnetic flux tubes attached would be classified as anyons according to winding phases but as fermions according to his exclusion principle.

In this paper, we resolve the apparent conflict between these two definitions of fractional statistics (see Fig. 1). The argument consists of several parts. First, we show that in the one-dimensional system obeying a fractional exclusion principle, the HSM, an analog of a winding phase, i.e., a statistical phase acquired by the wave function as the anyons go through each other, exists. The conflict with the topological considerations explained above is circumvented in that the crossing of the spinons occurs in one direction only. The statistical phase of $\pi/2$ acquired by the wave function as the spinons cross manifests itself in a fractional shift for the spacings of the single-spinon momenta.

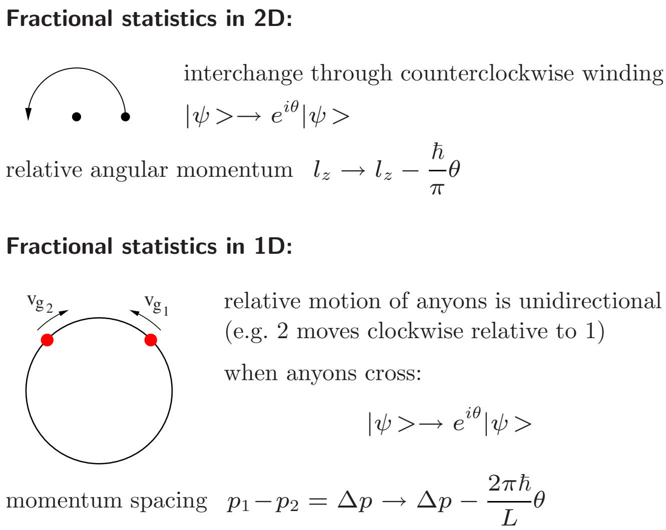

FIG. 1. (Color online) Fractional statistics in 2D and in 1D.

Second, we show that a fractional shift for the momentum spacings, and hence a statistical phase of $\pi/2$ acquired by the wave function, also exists for the holons in the Kuramoto-Yokoyama model (KYM),[11] the supersymmetrically extended HSM allowing for itinerant holes. The holons are hence half fermions, a conclusion reached previously by Ha and Haldane[12] using the asymptotic Bethe ansatz (ABA), by Kuramoto and Kato[13] from thermodynamics, and by Arikawa *et al.*[14] from the electron addition spectral function of the model. Since the N localized single-holon states of the KYM are orthogonal, however, they appear to be fermions according to Haldane's exclusion statistics. As a resolution of the conflict, we propose that the exclusion principle yields the correct statistics only when applied to energy eigenstates of a given model. The statistics obtained via the exclusion principle is then always consistent with the statistics manifest in physical quantities, i.e., the momentum spacings in 1D or the quantization of the relative angular momenta in 2D.

Finally, we argue that the picture we propose—crossings in only one direction, statistical phases acquired by the wave function as anyons go through each, fractionally spaced single anyon momenta—holds for one-dimensional anyons in general.

II. FIRST EXAMPLE OF IDEAL ANYONS: TWO SPINONS IN THE HALDANE-SHASTRY MODEL

The subtleties involved are best explained by looking closely at two-spinon and two-holon eigenstates of the KYM. The model is conveniently formulated by embedding the one-dimensional chain with periodic boundary conditions (PBCs) into the complex plane by mapping it onto the unit circle with the sites located at complex positions $\eta_\alpha = \exp(i\frac{2\pi}{N}\alpha)$, where N is the number of sites and $\alpha = 1, \ldots, N$. Each site can be occupied either by an up-spin or down-spin electron or a hole (i.e., the site is empty). The Hamiltonian is given by

$$H_{\mathrm{KY}} = -\frac{2\pi^2}{N^2}\sum_{\alpha\neq\beta}^{N}\frac{P_{\alpha\beta}}{|\eta_\alpha - \eta_\beta|^2}, \tag{1}$$

where the graded permutation operator $P_{\alpha\beta}$ exchanges particles on sites η_α and η_β, multiplied by a minus sign if both particles are fermions (i.e., neither of them a hole). In the absence of holes, Eq. (1) reduces to the HSM, which possesses the exact ground state,

$$\Psi_0[z_i] = \prod_{i<j}^{M}(z_i - z_j)^2\prod_{i=1}^{M} z_i, \tag{2}$$

for N even, $M = N/2$, and $[z_i] \equiv (z_1, \ldots, z_M)$. The z_i's denote the positions of the up spins. The corresponding state vector is given by

$$|\Psi_0\rangle = \sum_{\{z_1,\ldots,z_M\}}\Psi_0(z_1,\ldots,z_M)S_{z_1}^{+}\ldots S_{z_M}^{+}|0_\downarrow\rangle, \tag{3}$$

where the sum extends over all possible ways to distribute the positions z_i of the up spins over the N sites. $|0_\downarrow\rangle = \prod_\alpha c_{\alpha\downarrow}^{\dagger}|0\rangle$ denotes the fully spin-polarized reference state.

The greatly simplifying feature of the HSM (and the KYM) is that the spinons (and the holons) are free in the sense that they only "interact" through their half-Fermi statistics.[15–17] This renders these models particularly suited for our studies.

Let us now turn to the two-spinon eigenstates. A momentum basis for spin-polarized two-spinon states is given by

$$\Psi_{mn}[z_i] = \sum_{\alpha,\beta}^{N}(\bar{\eta}_\alpha)^m(\bar{\eta}_\beta)^n\prod_{i=1}^{M}(\eta_\alpha - z_i)(\eta_\beta - z_i)\Psi_0[z_i], \tag{4}$$

where $M = (N-2)/2$ and $M \geq m \geq n \geq 0$. For m or n outside this range, Ψ_{mn} vanishes identically, reflecting the overcompleteness of the position space basis. Acting with Eq. (1) on Eq. (4) yields[18]

$$H_{\mathrm{KY}}|\Psi_{mn}\rangle = E_{mn}|\Psi_{mn}\rangle + \sum_{l=1}^{l_{\max}} V_l^{mn}|\Psi_{m+l,n-l}\rangle, \tag{5}$$

with $l_{\max} = \min(M-m, n)$, $V_l^{mn} = -\frac{2\pi^2}{N^2}(m-n+2l)$, and

$$E_{mn} = E_0 + \epsilon(q_m) + \epsilon(q_n). \tag{6}$$

Here, $E_0 = -\frac{\pi^2}{4N}$ is the ground-state energy,

$$\epsilon(q) = \frac{1}{2}q(\pi - q) + \frac{\pi^2}{8N^2}, \tag{7}$$

and we have identified the single-spinon momenta for $m \geq n$ according to

$$q_m = \pi - \frac{2\pi}{N}\left(m + \frac{1}{2} + s\right), \quad q_n = \pi - \frac{2\pi}{N}\left(n + \frac{1}{2} - s\right), \tag{8}$$

with a *statistical shift* $s = 1/4$. Since the "scattering" of the nonorthogonal basis states $|\Psi_{mn}\rangle$ in Eq. (5) only occurs in one direction, increasing $m-n$ while keeping $m+n$ fixed, the eigenstates of H_{KY} have energy eigenvalues E_{mn} and are of the form

$$|\Phi_{mn}\rangle = \sum_{l=0}^{l_M} a_l^{mn} |\Psi_{m+l,n-l}\rangle. \tag{9}$$

A recursion relation for the coefficients a_l^{mn} is readily obtained from Eq. (5).

The relevant feature for our present purposes is the shift s in the single-spinon momenta (8), which we will elaborate on now. The state [Eq. (4)] tells us unambiguously that the sum of both spinon momenta is given by

$$e^{i(q_m+q_n)} = \frac{\Psi_{mn}[\eta_1 z_i]}{\Psi_{mn}[z_i]} \frac{\Psi_0[z_i]}{\Psi_0[\eta_1 z_i]}, \tag{10}$$

with $[\eta_1 z] \equiv (\eta_1 z_1, \ldots, \eta_1 z_M)$, which implies $q_m + q_n = 2\pi - \frac{2\pi}{N}(m+n+1)$, and hence Eq. (8). The shift s is determined by demanding that the excitation energy [Eq. (6)] of the two-spinon state is a sum of single-spinon energies, which in turn is required for the explicit solution here to be consistent with the ABA results.[15–17]

The appearance of this shift, which decreases the momentum q_m of spinon 1 and increases momentum q_n of spinon 2, is somewhat surprising given that the basis states (4) are constructed symmetrically with regard to interchanges of m and n. To understand this asymmetry, note that $M \geq m \geq n \geq 0$ implies $0 < q_m < q_n < \pi$. Dispersion (7) implies that the group velocity of the spinons is given by

$$v_{\mathrm{g}}(q) = \partial_q \epsilon(q) = \frac{\pi}{2} - q, \tag{11}$$

which in turn implies that $v_{\mathrm{g}}(q_m) > v_{\mathrm{g}}(q_n)$. The physical significance of this result can hardly be overstated. It means that the *relative motion* of spinon 1 (with q_m) with respect to spinon 2 (with q_n) is *always counterclockwise* on the unit circle. Then, however, the shifts in the individual spinon momenta can be explained by simply assuming that the two-spinon state acquires a statistical phase $\theta = 2\pi s$ whenever the spinons pass through each other. This phase implies that q_m is shifted by $-\frac{2\pi}{N}s$ since we have to translate spinon 1 counterclockwise through spinon 2 and hence counterclockwise around the unit circle when obtaining the allowed values for q_m from the PBCs. Similarly, q_n is shifted by $+\frac{2\pi}{N}s$ since we have to translate spinon 2 clockwise through spinon 1 and hence clockwise around the unit circle when obtaining the quantization of q_n. [The fact that the "bare" ($s=0$) values for q_m and q_n are quantized as $\frac{2\pi}{N}(\frac{1}{2} + \text{integer})$ is related to the bosonic representation of the bare spinons. If we had chosen a fermionic representation, they would be quantized as $\frac{2\pi}{N} \times \text{integer}$.]

That the crossing of the spinons occurs only in one direction is not just a peculiarity but a necessary requirement for fractional statistics to exist in 1D at all. If the spinons could cross in both directions, the fact that paths interchanging them twice (i.e., once in each direction) are topologically equivalent to paths not interchanging them at all would imply $2\theta = 0 \bmod 2\pi$ for the statistical phase, i.e., only allow for the familiar choices of bosons or fermions. With the scattering occurring in only one direction, arbitrary values for θ are possible. The one-dimensional anyons neither break the time-reversal (T) symmetry nor parity (P).

III. SECOND EXAMPLE OF IDEAL ANYONS: TWO HOLONS IN THE KURAMOTO-YOKOYAMA MODEL

We now turn to the two-holon eigenstates of the KYM,[19] which are highly instructive with regard to Haldane's exclusion principle as a definition of fractional statistics. A momentum basis for two-holon states is given by

$$\Psi_{mn}^{\mathrm{ho}}[z_i, h_j] = \phi_{mn}(h_1, h_2) \prod_{i=1}^{M} (h_1 - z_i)(h_2 - z_i) \Psi_0[z_i], \tag{12}$$

where $M = (N-2)/2$ and $[z_i, h_j] \equiv (z_1, \ldots, z_M; h_1, h_2)$. The z_i's denote the positions of the up spins again and h_1 and h_2 the positions of the holes. $\phi_{mn}(h_1, h_2)$ is an internal holon-holon wave function, which has to be homogeneous and antisymmetric under interchange of h_1 and h_2. Using an educated guess,

$$\phi_{mn}(h_1, h_2) = (h_1 - h_2)(h_1^m h_2^n + h_1^n h_2^m), \tag{13}$$

we obtain

$$H_{\mathrm{KY}}^{\mathrm{ho}} |\Psi_{mn}^{\mathrm{ho}}\rangle = E_{mn}^{\mathrm{ho}} |\Psi_{mn}^{\mathrm{ho}}\rangle + \sum_{l=1}^{l_{\max}} V_l^{mn} |\Psi_{m-l,n+l}^{\mathrm{ho}}\rangle \tag{14}$$

for $0 \leq n \leq m \leq M+1$. If this condition is violated, the basis states $|\Psi_{mn}^{\mathrm{ho}}\rangle$ do not vanish identically, but it is not possible to construct eigenstates from them. In Eq. (14), $l_{\max}$ is the largest integer $l \leq \frac{m-n}{2}$, $V_l^{mn} = \frac{2\pi^2}{N^2}(m-n)$, and

$$E_{mn}^{\mathrm{ho}} = E_0 + \epsilon^{\mathrm{ho}}(p_m) + \epsilon^{\mathrm{ho}}(p_n). \tag{15}$$

The single-holon energies are given by

$$\epsilon^{\mathrm{ho}}(p) = \frac{1}{2} p(\pi + p) - \frac{\pi^2}{8N^2}, \tag{16}$$

and we have identified the single-holon momenta for $m \geq n$ according to

$$p_m = -\pi + \frac{2\pi}{N}(m+s), \quad p_n = -\pi + \frac{2\pi}{N}(n-s), \tag{17}$$

with $s = 1/4$. The scattering occurs again only in one direction, this time decreasing $m-n$ while keeping $m+n$ fixed, which implies both that the basis states $|\Psi_{mn}^{\mathrm{ho}}\rangle$ are not orthogonal and that the two-holon eigenstates of H_{KY} have energy eigenvalues E_{mn}^{ho}. The statistical shift s is once again determined by demanding that the holons are free, which in turn is required by consistency with the ABA solutions.[16]

The momenta are again limited to about half of the Brillouin zone, $-\pi - \frac{\pi}{2N} \leq p_n < p_m \leq \frac{\pi}{2N}$. With the holon group velocity,

$$v_{\mathrm{g}}^{\mathrm{ho}}(p) = \partial_p \epsilon^{\mathrm{ho}}(p) = \frac{\pi}{2} + p, \tag{18}$$

we obtain $v_g^{ho}(p_m) > v_g^{ho}(p_n)$. The crossing of the holons occurs again only in one direction, and the momentum shifts as well as the half-Fermi statistics emerges as in the case of the spinons, except that the state now acquires the phase $\theta = -2\pi s$, with the result that the momentum p_m of the holon with the larger group velocity $v_g^{ho}(p_m)$ is shifted by $+\frac{2\pi}{N}s$ and p_n is shifted by $-\frac{2\pi}{N}s$. Physically, this reversal in the sign reflects that the holon is created by annihilation of an electron at a spinon site, i.e., by removing a fermion from a half fermion. The spacing between p_m and p_n, however, is quantized as for the spinons above. Note that the hard-core constraint of the holons is irrelevant here.

Let us now reconcile this result with the exclusion principle. As mentioned, the hard-core condition for holons effects that they are fermions according to Haldane's exclusion principle applied to states localized in position space. When applied to exact eigenstates of the model, however, the result is different. Since the creation of two holons decreases the number of up or down spins in the uniform liquid M by 1, the number of single-holon states (labeled by m or n above) available for additional holons decreases by 1. This implies half-Fermi statistics and is consistent with the momentum spacings as well as previous work.[12–14]

The exclusion principle hence yields the correct statistics only if applied to eigenstates of the model. The wave function for localized holons is really a superposition of a holon state [onto which we project in Eq. (12)] and a holon surrounded by an incoherent spinon cloud in a singlet configuration.

IV. GENERAL VALIDITY OF THE CONCLUSIONS

So far, our discussion has been limited to a particular model. The conclusions, however, hold in general. As noted above, the KYM is special in that the spinon and holon excitations are free. The single spinon and holon momenta are hence good quantum numbers. The eigenstates of the model can be labeled in terms of these momenta, which we have shown to be fractionally spaced. Any other model of a one-dimensional spin chain can be described as a KYM supplemented by additional terms, which give rise to an interaction between the spinons and holons. This interaction will scatter the basis states of free spinons and holons, the eigenstates of the KYM, into each other. The eigenstates of the interacting model will hence be superpositions of states with different single spinon and holon momenta, all of which, however, will be fractionally spaced. In other words, the fractional shifts in Eqs. (8) and (17) [and also Eqs. (19) and (20) below] will still be good quantum numbers, while the integers n and m will no longer be good quantum numbers. [Of course, if we turn on an interaction such that the quantum numbers of the excitations change (e.g., from spinons to spin flips), the Hilbert space will change and the argument will break down. This, however, is beyond the point as we make a statement about the statistics of spinons and not the spin flips.]

This argument shows that whenever we have spinons and holons in a one-dimensional spin chain, we have fractionally spaced single-particle momenta as a consequence of their fractional statistics. Is it reasonable to assume that this picture holds for anyons in 1D in general? We believe there are very good reasons to do so. First, spinons and holons are the only known examples of anyons in 1D. This picture hence holds for all examples of 1D systems with fractional statistics. Second, the picture resolves a profound conflict, as topology precludes the existence of one-dimensional anyons in a conventional framework of indistinguishable particles. The conflict is circumvented here in that the anyons become distinguishable through their dynamics and cross in one direction only. If the picture we propose here were not of general validity, another resolution to this conflict would have to exist. This does not appear to be the case.

V. SUMMARY

We conclude with a summary. We propose that the statistics of identical particles is always reflected in the quantization condition of an observable quantity. For anyons with statistical parameter θ in 2D, the kinematical relative angular momentum between two anyons is quantized as[5]

$$l_z = \hbar\left(-\frac{\theta}{\pi} + 2m\right), \tag{19}$$

where $-\pi < \theta \leq \pi$ and m is integer.

For anyons with statistical parameter θ in a one-dimensional system with length L and periodic boundary conditions—and this is the central message of this paper—the allowed values for the spacings between the kinematical (linear) momenta are quantized as

$$p_{i+1} - p_i = \Delta p = \frac{2\pi\hbar}{L}\left(\frac{|\theta|}{\pi} + n\right) \tag{20}$$

for $p_{i+1} - p_i \geq 0$, where $-\pi \leq \theta \leq \pi$ and n is a non-negative integer. The spacing condition (20) holds for many-anyon states with single-particle momenta $p_1 \leq p_2 \leq \ldots \leq p_N$ in any interval $p_i \in \mathcal{I}$ provided that the anyon group velocity $v_g(p) = \partial_p \epsilon(p)$ is a strictly increasing ($\theta < 0$) or decreasing ($\theta > 0$) function of p in this interval. This condition is required for the anyons to cross in one direction only. In an interacting many-particle system, the quantum numbers m and n in Eqs. (19) and (20) are not expected to be good quantum numbers. The fractional shifts $-\theta/\pi$ and $|\theta|/\pi$, however, are topological invariants.

Note that Eqs. (19) and (20) hold only between the *physical* or *kinematical* statistics of the anyons and the kinematical angular or linear momenta, as canonical momenta are gauge dependent. In particular, one may change the canonical momenta while simultaneously changing the canonical statistics of the fields (i.e., the statistics imposed when canonically quantizing the fields) used to describe the anyons via a "singular gauge transformation." The canonical statistics may either be bosonic, as in the case of the spinons in the analysis above, or fermionic, as in the case of the holons above.

Our analysis further demonstrates that particular care must be exercised when defining statistics using Haldane's exclusion principle. The fact that it gives the correct result for the statistics of holons in the KYM when applied to eigenstates of the model but an incorrect result when applied to holon states localized in position space leads us to conjecture that in general, the exclusion principle yields correct results only when applied to eigenstates of a given model.

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