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HEURISTIC PRINCIPLE FOR QUANTIZED HALL STATES

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It is argued that the incompressibility of fractional quantized Hall states, and the qualitative form of their wavefunction, can be understood by an argument based on adiabatic localization of magnetic flux. The hierarchy of allowed filling fractions is derived in a very simple way, that makes no reference to successive condensations of quasiparticles.

1. Adiabatic Principle

There can be little doubt that the leading idea of the theory of the fractional quantized Hall effect, the existence of incompressible quantum liquids at favorable filling fractions, is correct.[1,2] Also, it is probably true that the hierarchical trial wavefunctions which have been suggested[3–7] to describe these states capture the essential physics, and have non-zero overlap with the true ground state wavefunctions.[8] For the first level of the hierarchy, if the repulsive interaction potential is taken to have a simple infinitely short-ranged form, the Laughlin $1/m$ wavefunctions are in fact known to be exact.[3,9]

Nevertheless one may feel a certain dissatisfaction with the existing theory: not that it is wrong, but that its correctness should be more obvious. The fundamental qualitative properties of the states: the existence of specially favored filling fractions and the incompressibility of the exact ground states at these filling fractions, deserve to be understood in a similarly fundamental way. Instead, in the present state of theory, the existence of favored filling fractions is "explained" by the fact that certain *ad hoc* construction of trial wavefunctions only work at these fractions, and the incompressibility is "explained" by use of a clever mapping of these trial wavefunction onto a two-dimensional Coulomb gas, where it is seen as a consequence of screening in the plasma phase.

A notable recent attempt to provide a better heuristic understanding is the work of Jain and his collaborators.[10] He sees the essence of the phenomenon in

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the formation of composite particles, consisting of several electrons and flux tubes. The motivation for, and the appeal of, such an idea is easily seen upon contemplating the form of the Laughlin $1/m$ function:

$$\psi_m = \prod_{i<j} (z_i - z_j)^m \prod_i e^{-1/4l^2|z_i|^2} . \tag{1.1}$$

Here of course m is an odd integer, and $l^2 = \frac{1}{eB}$ is the square of the magnetic length. Now if we imagine that a flux tube carrying the unit flux $\frac{2\pi}{e}$, in the direction *opposite* to that of the external B, at z_i imparts a factor $z_i - z_j$ in the wavefunction for the position of every electron z_j, then we may imagine building up this wavefunction for $N+1$ particles from the one for N by inserting $m-1$ unit flux tubes at the position of the newly inserted electron. This operation will, together with the single powers of $z_i - z_j$ arising from Fermi statistics, generate the first factor on the right-hand side of (1.1).

Since it is known that changes in quantum statistics[11,12] are central to the fractional quantized Hall effect,[5,13] and that changes in quantum statistics can be implemented by attaching fictitious flux tubes, an alternative phrasing of the same observation is suggestive. One may say that the electrons have become *superfermions*: as one winds around another, one obtains not only a single minus sign — i.e., $e^{i\pi}$, phase π — but the larger phase $2\pi m$.

Despite its intuitive appeal the basic concept of this picture, in either formulation, is a shaky one. Strictly speaking, point flux tubes carrying any integral multiple of the basic flux unti are negligible: they can be gauged away. Similarly, the notion of super-fermions is quite peculiar, because in the end there is still just a minus sign when one particle winds around another: $e^{i\pi m} = e^{i\pi}$.

However, if we consider the attachment of several quanta of flux, or the acquisition of super-fermi statistics, as *processes* rather than as completed acts, then we can make sense of these notions. Thus we are led to propose, as a heuristic principle, that quantized Hall states – that is spatially uniform incompressible quantum liquids – result from the adiabatic localization of flux, or equivalently from the adiabatic change of particle statistics, starting from other quantized Hall states.

Why is this reasonable? An incompressible quantum liquid ground state is characterized by an energy gap, and thus it is sensible to consider its adiabatic evolution under a perturbation.[14] Furthermore, the perturbation we have in mind here — of localizing magnetic flux on the particles — is spatially uniform, and should lead us from one homogeneous incompressible state to another. Thus as long as the gap does not collapse — and we see no reason to expect that it will, at least for sufficiently small rearrangements of flux — we expect that adiabatic evolution leads from one incompressible, uniform ground state to another.

A more precise mathematical formulation can be given in terms of the Chern-Simons construction.[15] We consider a family of Hamiltonians

$$H = \psi^* \frac{(i\partial - eA^\eta - q^\eta a)^2}{2m} \psi + \frac{\mu}{2} \varepsilon^{\alpha\beta\gamma} a_\alpha f_{\beta\gamma} + V , \tag{1.2}$$

where ψ is the second quantized electron field operator, A is the vector potential corresponding to the external magnetic field B, $A^\eta \equiv (1+\eta)A$, a is the fictitious field operator and $(q^\eta)^2 \equiv \eta \frac{\mu e B}{\rho}$. V contains the (repulsive) electron-electron interactions, which for simplicity we imagine do not vary with η. Then our principle is, that if we start from a quantized Hall state and slowly vary η, it is reasonable to expect that the adiabatic evolution of this state leads to other quantized Hall states.

Now as we evolve η we shall have, according to our heuristic prinicple, a variety of quantized Hall states for electrons with flux attached, i.e., *anyons*. It is of some interest to construct quantized Hall states of anyons, since it is quite possible that in suitable circumstances the quasiparticles of $(2+1)$-dimensional materials are charged anyons. This line of thought will be pursued further elsewhere.[16] In the present note, however, we will discuss the relevance of our heuristic principle to the ordinary fractional quantized Hall effect of electrons.

The basic point is that *if we can carry the adiabatic evolution far enough, so that at the end an even number of flux quanta are localized on the electrons, then we arrive once more at fermions*, and thus at another quantized Hall state suitable to describe ordinary electrons.

As an application of this idea, let us consider how it works for the simplest of all quantized Hall states — a single filled Landau level. If we localize $2n$ flux units on each electron, i.e., $\eta = 2n$, then the final magnetic field is $B^\eta = (2n+1)B$. Since the density was such as to just fill a single Landau level for the original field B, we see that in this way we arrive at quantized Hall states with filling fractions $\frac{1}{2n+1}$. While of course these filling fractions have long been strongly suspected to support quantized Hall states, we believe it is of considerable methodological importance that the present argument is (in cases where it applies at all) exact, intrinsically non-variational, and independent of the detailed form of V; and that it provides a simple heuristic explanation both of why exactly these fractions are relevant and of the incompressibility of the states.

The local structure of the adiabatically evolved wavefunction — that is, its form when two electrons closely approach one another — is very plausibly of the same form as in Laughlin's *ansatz*, since this is precisely the form reached for two-body adiabatic evolution.

2. Possible Generalizations

A generalization of the previous construction,[10] is to begin the evolution with a wider class of known quantized Hall states. Let us begin the evolution with real flux density B_i and end with B_f, in the process localizing $2n$ flux units on each electron. Then we have

$$\frac{eB_i}{2\pi} = \frac{eB_f}{2\pi} - 2n\rho \ , \tag{2.1}$$

or in terms of filling fractions

$$\frac{1}{\nu_i} = \pm\frac{1}{\nu_f} - 2n \ , \tag{2.2}$$

where the sign reflects the relative orientation of B_i and B_f. If we take one of the most securely known quantized Hall states, namely r filled Landau levels, as the initial configuration, then we find

$$\nu_f = \frac{r}{2nr \pm 1} \ . \tag{2.3}$$

It is noteworthy that this series of filling fractions, for the simplest possibility $n = 1$, includes the most prominently observed ones. In the hierarchical scheme, these fractions appear at level r, and the corresponding states are quite complicated for large r.

Nothing new is obtained by iterating this process, i.e., by beginning from one of the fractions in Eq. (2.3) and adiabatically localizing flux until another fermion state is found. This closure property follows immediately from Eq. (2.2). Thus the ordinary hierarchical picture, leading to other fractions, is not included in our scheme so far.

[Another weakness is that if the final real flux is opposite in direction to the initial real flux, then at some intermediate stage in the adiabatic evolution it must have passed through zero. When this occurs the effective anyon system is generally expected to be compressible. It therefore supports a gapless sound mode which may be the only gapless excitation (then we have an anyon superfluid), or may dissolve into a continuum of quasiparticle-quasihole pairs (then we have an anyon metal). In either case, the hypothesis of adiabatic evolution becomes questionable. The situation is far from hopeless, since long-range Coulomb interactions will produce a gap (Higgs mechanism), and even in the absence of a gap sound waves are not easily excited by uniform perturbations.]

Other fractions appear when we avail ourselves of particle-hole conjugation.[4,6] States obtained by particle-hole conjugation become an excellent approximation, when the magnetic field is large and the interactions small. The symmetry becomes exact when transitions to intermediate state out of the lowest Landau level are neglected. The filling fraction is, of course, modified by $\nu \to 1 - \nu$.

We now indicate how the two operations, adiabatic localization and particle-hole conjugation, together generate the complete hierarchy of filling fractions.[17] It is very convenient to introduce the parameter $\alpha \equiv \frac{1}{\nu} - 1$, in terms of which these operations read $\alpha \to \pm\alpha + 2n$, and $\alpha \to 1/\alpha$.

Now we wish to show that any rational ν with an odd denominator in reduced terms, or equivalently any α of the form $\frac{p}{q}$ with one of p, q even and the other odd, can be obtained by these operations, starting with a single filled Landau level ($\alpha = 0$). Since the operations are reversible, this problem is the same as showing that any α of this form can be reduced to zero. If $p > q$, we can obviously write $p = 2np \pm r$ with n and r integers and $0 \le r < q$. The second inequality is always proper, because p and q have opposite parity. By applying the operation $\alpha \to \pm\alpha + 2n$ we can therefore reduce α to the form $\frac{r}{q}$. Now if $r = 0$, we are done. Otherwise, we may apply the operation $\alpha \to 1/\alpha$ to make $\alpha = \frac{q}{r} > 1$, and iterate the whole procedure. Since the integers in the fractions are ever decreasing, after a finite number of steps we arrive at $\alpha = 0$, as was to be proved. Of course if the initial α is < 1, the first step is to be omitted.

3. Filling Fractions and States: Some General Considerations

In the previous section we discussed two quite distinct methods for generating states that were plausibly incompressible fluids, one of which involved crushing several filled Landau levels, the other of which involved pirouetting within the lowest Landau level. At the fractions appearing on the right-hand side of Eq. (2.3), we appear to have two quite different constructions leading to the same filling fractions.

A close look at the quantum numbers of quasiparticles in these states shows that they are in fact different in general, despite the fact that the states occupy the same filling fraction. Should this disturb us?

No. There are many other examples where several distinctly different physical states may, under different circumstances — that is, for different forms of the interparticle interaction — be the ground state *for the same filling fraction.* The most elementary example, of course, is that the free fermion gas is metallic at any filling fraction, say $\nu = 1/3$, whereas repulsive interactions can drive it into the fractional quantized Hall state.

A less trivial, but clear-cut and instructive example is the following, developed in conversation with X. G. Wen. Let us suppose that the potential is such that groups of three electrons form tightly bound triples, with the triples mutually repelling. Then we have effectively a gas of charge $e^* = 3e$ particles. Filling fraction $\nu^* = 1/9$ for these triples corresponds to filling fraction $\nu = 1$ for the original electrons. If the quasiholes of the $\nu^* = 1/9$ states carry the expected charge

— $e^*/9$, then we have produced (in terms of the original electrons) a $\nu = 1$ state supporting charge — $e/3$ quasiholes, which if no crossings have occurred will be the incompressible ground state. Thus even at the filling fraction corresponding to a single completely filled Landau level, if the interactions are strong and of the right form, exotic incompressible states can arise, including ones that support fractionally charged excitations.

Let us consider the question of uniqueness more generally, in the context of the hierarchical construction of the previous section. Our procedure for reducing α, although the most straightforward and systematic, was not unique. We applied the inversion operation $\alpha \to 1/\alpha$ only when α was between 0 and 1, but it is well-defined as a formal transformation for any non-zero value of α, and makes perfectly good physical sense for any positive α, i.e., $\nu < 1$. If we perform inversion operations of this kind at intermediate stages, clearly we can produce an infinite number of alternative procedures leading from a single filled Landau level to any given filling fraction.

4. Comments

1. Our operations generate an easily characterized matrix group of transformations on $\alpha = p/q$, when the pair p, q is regarded as a two-dimensional vector. In fact, we find that precisely those matrices of the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with a, b, c, d integers satisfying $a, d \equiv \pm 1 \pmod 2$ and $b, c \equiv 0 \pmod 2$, or the same conditions with a, d and b, c interchanged, are generated. This group is essentially what the mathematicians call the principle congruence group[18] at level 2.

2. An amusing special case of the construction in the previous section concerns an integral number n of completely filled Landau levels. Following the canonical reduction procedure one expresses this as a continued fraction with essentially n terms, e.g.:

$$4 = \cfrac{1}{1 - \cfrac{1}{2 - \cfrac{1}{2 - \cfrac{1}{2}}}}\,.$$

This appears contrived. A reasonable alternative standpoint is to declare that any integer filling fraction is a plausible starting point for constructing incompressible states without further analysis, since n filled Landau levels for free fermions has this property. If we adopt this standpoint (which, in a slightly different language, seems to be a key element of Jain's work), we can drastically shorten the path to some filling fractions, e.g., to $\nu = \frac{r}{2r \pm 1}$ as discussed before.

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