Parent Hamiltonian for the non-Abelian chiral spin liquid

Martin Greiter,1 Darrell F. Schroeter,2 and Ronny Thomale1
1Institute for Theoretical Physics, University of Würzburg, Am Hubland, 97074 Würzburg, Germany
2Department of Physics, Reed College, Portland, Oregon 97202, USA

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We construct a parent Hamiltonian for the family of non-Abelian chiral spin liquids proposed recently by two of us [Phys. Rev. Lett. 102, 207203 (2009)] which includes the Abelian chiral spin liquid proposed by Kalmeyer and Laughlin as the special case $s = \frac{1}{2}$. As we use a circular disk geometry with an open boundary, both the annihilation operators we identify and the Hamiltonians we construct from these are exact only in the thermodynamic limit.

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Introduction. The field of two-dimensional quantum spin liquids [1–15] is witnessing a renaissance of interest in present days [16–20]. For one thing, due to advances in the computer facilities available, evidence for spin liquid states in a range of models is accumulating [21,22]. At the same time, spin liquids constitute the most intricate, and in general probably least understood, examples of topological phases [23–28], which themselves establish another vividly studied branch of condensed matter physics [29–31]. If a complete description of the electronic states in the two-dimensional (2D) CuO planes of high $T_c$ superconductors [32] ever emerges, the theory is least understood, examples of topological phases [23–28], (TD) limit even if we impose PBCs.

Chiral spin liquid states. The conceptually simplest way to construct the non-Abelian chiral spin liquid (NACSL) state [13] with spin $s$ is to combine $2s$ identical copies of Abelian CSL states with spin $\frac{1}{2}$, and project the spin on each site onto spin $s$,

$$\frac{1}{2} \otimes \frac{1}{2} \otimes \cdots \otimes \frac{1}{2} = s \oplus (2s - 1) \oplus \cdots \oplus \frac{1}{2}$$

The projection onto the completely symmetric representation can be carried out conveniently using Schwinger bosons [7,47]. For a circular droplet with open boundary conditions occupying $N$ sites on a triangular or square lattice, the Abelian CSL state takes the form

$$|\psi_0^{\text{KL}}\rangle = \sum_{\{z_1,\ldots,z_M\}} \psi_0^{\text{KL}}(z_1,\ldots,z_M) S_{z_1}^+ \cdots S_{z_M}^+ |\downarrow\cdots\downarrow\rangle$$

where

$$\psi_0^{\text{KL}}[a^\dagger,b^\dagger]|0\rangle = \prod_{i<j} (z_i - z_j)^2 \prod_{i=1}^M G(z_i) e^{-\frac{1}{2}|z|^2}$$

is a bosonic quantum Hall state in the complex “particle” coordinates $z_i = x_i + iy_i$, supplemented by a gauge factor $G(z_i)$, $M = \frac{N}{2}$, $a^\dagger$ and $b^\dagger$ are Schwinger boson creation operators [7,47,48], and the $w_k$’s are those lattice sites which are not occupied by any of the $z_i$’s. In this notation, we can write the spin $s$ state obtained by the projection as

$$|\psi_0^s\rangle = (\psi_0^{\text{KL}}[a^\dagger,b^\dagger])^{2s}|0\rangle$$

The lattice may be anisotropic; we have chosen the lattice constants such that the area of the unit cell spanned by the primitive lattice vectors is set to $2\pi$. For a triangular or square lattice with lattice positions given by $\eta_{n,m} = na + mb$, where $a$ and $b$
and \( b \) are the primitive lattice vectors in the complex plane and \( n \) and \( m \) are integers, the gauge phases are simply \( G(\eta_{a,m}) = (-1)^{\alpha_{a}+\beta_{a}+1} \) [46,49].

The NACSL state can alternatively be written as

\[
|\psi_0^s\rangle = \sum_{\{z_1, \ldots, z_N\}} \psi_0^s(z_1, \ldots, z_N) \tilde{\mathbf{S}}^+_1 \cdots \tilde{\mathbf{S}}^+_N |s\rangle_N, \tag{4}
\]

where \( |s\rangle_N \equiv \otimes_{a=1}^N |s, -s\rangle_a \) is the “vacuum” state in which all the spins are maximally polarized in the negative \( \hat{z} \) direction, and \( \tilde{\mathbf{S}}^+ \) are renormalized spin flip operators which satisfy

\[
\frac{1}{\sqrt{(2\pi)^N}}(a^\dagger)^n(b^\dagger)^{2s-n}|0\rangle = (\tilde{\mathbf{S}}^+)^n|s, -s\rangle. \tag{5}
\]

In a basis in which \( S^z \) is diagonal, we may write

\[
\tilde{\mathbf{S}}^+ = \frac{1}{s - S^z + 1} \mathbf{S}^+. \tag{6}
\]

Note that Eq. (5) implies

\[
S^z(\tilde{\mathbf{S}}^+)^n|s, -s\rangle = n(\tilde{\mathbf{S}}^+)^{n-1}|s, -s\rangle. \tag{7}
\]

The wave functions for the spin \( s \) state (3) are then effectively given by bosonic Read-Rezayi states [42] for renormalized spin flips,

\[
\psi_0^s[z] = \sum_{m=1}^{2s} \prod_{i<j}^{mM} (z_i - z_j)^2 \prod_{i=1}^{sN} G(z_i) e^{-\frac{1}{2}|z_i|^2}, \tag{8}
\]

which we understand to be completely symmetrized over the “particle” coordinates \( z_i \). For \( s = 1 \), they take the form of a Moore-Read state [35,36]

\[
\psi_0^{s=1}[z] = \text{Pf} \left( \frac{1}{z_i - z_j} \right) \prod_{i<j}^{N} (z_i - z_j) \prod_{i=1}^{sN} G(z_i) e^{-\frac{1}{2}|z_i|^2}. \tag{9}
\]

For the considerations below, it is convenient to write the state in the form

\[
|\psi_0^s\rangle = \left[ \sum_{\{z_1, \ldots, z_M\}} \psi_0^{KL}(z_1, \ldots, z_M) \tilde{\mathbf{S}}^+_1 \cdots \tilde{\mathbf{S}}^+_M \right]^{2s} |s\rangle_N. \tag{10}
\]

Since the Abelian KL CSL \( |\psi_0^{KL}\rangle \) is an exact spin singlet in the TD limit \( N \rightarrow \infty \), and is an approximate singlet for finite \( N \), the same holds for the NACSL \( |\psi_0^s\rangle \) as well. This follows from the construction of the Schwinger boson projection (3), but can also be verified directly using Perelomov’s identity [50,51]. The Abelian and non-Abelian CSL states trivially violate parity (P) and time reversal (T) symmetry, which would take \( z \rightarrow -z \).

**Ground state annihilation operators.** In the TD limit \( N \rightarrow \infty \), the NACSL ground states are annihilated by

\[
\Omega_{\beta}^s = \sum_{\beta \neq \alpha} \frac{1}{\eta_{\alpha} - \eta_{\beta}} (S^+_{\alpha})^2 S^-_{\beta}, \quad \Omega_{\alpha}^s |\psi_0^s\rangle = 0 \quad \forall \alpha, \tag{11}
\]

as we will verify now.

Let us consider the action of \( (S^-_{\alpha})^2 S^-_{\beta} \) on \( |\psi_0^s\rangle \) written in the form (10). Since \( \psi_0^{KL}(z_1, \ldots, z_M) \) vanishes whenever two arguments \( z_i \) coincide, one of the \( z_i \)’s in each of the \( 2s \) copies in (10) must equal \( \eta_{\alpha} \); since \( \psi_0^{KL}(z_1, \ldots, z_M) \) is symmetric under interchange of the \( z_i \)’s and we count each distinct configuration in the sums over \( \{z_1, \ldots, z_M\} \) only once, we may take \( z_1 = \eta_{\alpha} \).

Regarding the action of \( S^-_{\beta} \) on (10), we have to distinguish between configurations with \( n = 0,1,2,\ldots,2s \) renormalized spin flips \( \tilde{S}^+_{\beta} \) at site \( \beta \). Since the state is symmetric under interchange of the \( 2s \) copies, we may assume that the \( n \) spin flips are present in the first \( n \) copies, and account for the restriction through ordering by a combinatorial factor. This yields
states that any infinite lattice sum of

\[ \sum_{\{\tau_1, \ldots, \tau_M\}} \psi^0_{KL}(\eta_{u}, \eta_{v}, \tau_2, \ldots, \tau_M) \hat{S}^{+}_{\tau_1} \cdots \hat{S}^{+}_{\tau_M} \]

where we have used the Perelomov identity \([50,51]\) which states that any infinite lattice sum of \(e^{-iH_{\eta}}G(\eta)\) times any analytic function of \(\eta_{\beta}\) vanishes.

**Parent Hamiltonian.** A Hermitian, positive semidefinite, and translationally invariant operator which annihilates \(|\psi^0_0\rangle\) is given by

\[
\Gamma \equiv \sum_{\alpha=1}^{N} \Omega_{\alpha}^{+} \Omega_{\alpha}^{2} = \sum_{\alpha, \beta, \gamma} \omega_{\alpha\beta\gamma} (S^{+}_{\alpha})^{2} (S^{-}_{\beta})^{2} S^{-}_{\gamma},
\]

where

\[
\omega_{\alpha\beta\gamma} = \frac{1}{\eta_{\alpha} - \eta_{\beta}} \frac{1}{\eta_{\alpha} - \eta_{\gamma}}.
\]

This operator is not invariant under SU(2) spin rotations, but rather consists of a scalar, vector, and higher tensor components up to order \(4s + 2\). Since the NACSL states \(|\psi^0_0\rangle\) are spin singlets, and are annihilated by \(\Gamma\), all these tensor components must annihilate the state individually \([52]\). The scalar component of \(\Gamma\), which we denote as \(|\Gamma\rangle_0\), provides us with an SU(2) spin rotationally invariant parent Hamiltonian.

To obtain the projected operator \(|\Gamma\rangle_0\), we follow the method described in detail in Ref. \([52]\), and summarize here only the most important steps. With the tensor content of \(S^{+}_{\beta} S^{-}_{\gamma}\) given by

\[
S^{+}_{\beta} S^{-}_{\gamma} = \frac{2}{3} S^{+}_{\beta} S^{-}_{\gamma} - i(S^{-}_{\beta} \times S^{-}_{\gamma})^2 - \frac{1}{\sqrt{6}} T^{0}_{\beta\gamma},
\]

where

\[
T^{0}_{\beta\gamma} = \frac{2}{3} (S^{+}_{\beta} S^{-}_{\gamma} - S^{-}_{\beta} S^{+}_{\gamma})
\]

\[
H^* = \sum_{a \neq \beta} \omega_{a\beta\gamma} \left[ s(s+1) + S_{a} S_{\beta} - \frac{(S_{a} S_{\beta})^2}{s(s+1)} \right] + \sum_{a, \beta, \gamma} \omega_{a\beta\gamma} \left[ (s+1) S^{+}_{\beta} S^{-}_{\gamma} - \frac{2s + 3}{2(s+1)} i S_{a} (S_{\beta} \times S_{\gamma}) - \frac{(S_{a} S_{\beta})(S_{a} S_{\gamma}) + (S_{a} S_{\gamma})(S_{a} S_{\beta})}{2(s+1)} \right].
\]

This is the \(m = 0\) component of the second order tensor, we only need to know the scalar, vector, and second order tensor components of \((S^{+}_{\alpha})^{2} (S^{-}_{\beta})^{2}\) in order to obtain the scalar component of \(\Gamma\). These are given by (see Sec. 5.3.2 of Ref. \([52]\))

\[
(S^{+}_{\alpha})^{2} (S^{-}_{\beta})^{2} = a_0 \left[ 1 + a S^{+}_{\alpha} b T^{0}_{\alpha\beta} + \text{higher orders} \right]
\]

where

\[
a_0 = \frac{(2s)^2}{2s + 1}, \quad a = \frac{3}{s + 1}, \quad b = \frac{\sqrt{6}}{2} \frac{5}{(s + 1)(2s + 3)}.
\]

The scalar component of \(\Gamma\) is hence given by

\[
|\Gamma\rangle_0 = a_0 \sum_{\alpha, \beta, \gamma} \omega_{a\alpha \beta\gamma} \left[ \frac{2}{3} S_{\beta} S_{\gamma} - \frac{ia}{3} S_{a} (S_{\beta} \times S_{\gamma}) - \frac{b}{\sqrt{6}} T^{0}_{\alpha\beta} T^{0}_{\alpha\gamma} \right]_0.
\]

With \(S_{\beta} \times S_{\gamma} = i S_{\beta}\) and (see Sec. 4.5.3 of Ref. \([52]\))

\[
5 \left[ T^{0}_{\alpha\beta} T^{0}_{\alpha\gamma} \right]_0 = -\frac{4}{3} S^{2}_{\alpha}(S_{\beta} S_{\gamma}) + 2 \delta_{\beta\gamma} S_{a} S_{\beta} + 2 [(S_{a} S_{\beta})(S_{a} S_{\gamma}) + (S_{a} S_{\gamma})(S_{a} S_{\beta})].
\]

we obtain the final parent Hamiltonian

\[
H^* = \sum_{a \neq \beta} \omega_{a\beta\gamma} \left[ s(s+1) + S_{a} S_{\beta} - \frac{(S_{a} S_{\beta})^2}{s(s+1)} \right] + \sum_{a, \beta, \gamma} \omega_{a\beta\gamma} \left[ (s+1) S^{+}_{\beta} S^{-}_{\gamma} - \frac{2s + 3}{2(s+1)} i S_{a} (S_{\beta} \times S_{\gamma}) - \frac{(S_{a} S_{\beta})(S_{a} S_{\gamma}) + (S_{a} S_{\gamma})(S_{a} S_{\beta})}{2(s+1)} \right].
\]
[It is related to Eq. (18) via \( \{ \Gamma \}_0 = 2\omega_0/(2s + 3) H^x \)]... Hamiltonian is approximately valid for any finite disk with \( N \) lattice sites, and becomes exact in the TD limit \( N \to \infty \), where \( H^x | \phi \rangle = 0 \). Note that the \( S_x(S_y \times S_y) \) term explicitly breaks P and T. (It would be highly desirable to identify a parent Hamiltonian which is P and T invariant, such that the ground states violate these symmetries spontaneously, but we have so far not succeeded in finding one.)

The special case \( s = \frac{1}{2} \). Since \( S_y^2 = 0 \) for \( s = \frac{1}{2} \), \( T_{a \beta}^m = 0 \) for all \( m \), and \( (T_{a \beta i}^m T_{a \beta j}^m)_{i j} = 0 \). This simplifies Eq. (18) significantly, and yields the parent Hamiltonian

\[
H^{x = \frac{1}{2}} = \sum_{a \neq \beta} \omega_{a \beta} \left[ \frac{3}{4} + S_a S_\beta \right] + \sum_{\alpha \neq \beta, \gamma \neq \beta} \omega_{a \beta \gamma} [S_\beta S_\gamma - i S_a (S_\beta \times S_\gamma)] .
\]  

(21)

[It is related to Eq. (18) via \( \{ \Gamma \}_0 = 2\omega_0/(2s + 3) H^{x = \frac{1}{2}} \).] In contrast to the earlier parent Hamiltonian proposed in Refs. [44] and [45] (SKTG) for the Abelian KL CSL (2) with periodic boundary conditions, Eq. (21) is not exact for finite \( N \). It is considerably simpler than the SKTG model, and, like Eq. (20), becomes exact in the TD limit.

Remarks on periodic boundary conditions. It is rather straightforward to formulate the model on a torus. For simplicity, we choose the lattice constant \( a \) real, and \( b \) such that the imaginary part \( \Im(b) > 0 \). We implement PBCs in both directions by identifying the sites \( z_i, z_i + L, \) and \( z_i + L \tau \), where \( L = n_1 a, L \tau = n_1 a + m_1 b, \) and \( \Im(\tau) > 0, \) \( n_1 \) and \( m_1 \) are positive integers such that the number of sites \( N = n_1 m_1 \) is even, and \( n_1 \) is an integer. We place the lattice sites at positions

\[
\eta_{n,m} = \left( n - \frac{n_1 - 1}{2} \right) a + \left( m - \frac{m_1 - 1}{2} \right) b ,
\]

(22)

with \( n = 0, 1, \ldots, n_1 - 1 \) and \( m = 0, 1, \ldots, m_1 - 1 \). Then the wave function of the NACSL (8) takes the form

\[
\psi_0^s[z] = \prod_{m=1}^{2s} \prod_{i,j = (m-1)M+1}^{mM} \vartheta^{\frac{1}{2}} \left( \frac{z_i - z_j}{L} \right) ^2 \times \prod_{\gamma=1}^{2} \vartheta^{\frac{1}{2}} \left( \frac{Z_{\gamma m} - Z_{\gamma,0}}{L} \right) ^2 \prod_{i=1}^{nN} G(z_i) e^{-\frac{1}{2} n^2} ,
\]

(23)

where \( \vartheta^{\frac{1}{2}}(z|\tau) \) is the odd Jacobi theta function [53], and

\[
Z_m = \sum_{i=(m-1)M+1}^{mM} z_i , \quad Z_{1,m} = -Z_{2,m} \quad \quad (24)
\]

are the center-of-mass coordinates and zeros, respectively. The latter can be chosen anywhere within the principal region bounded by the four points \( \frac{1}{2}(1 \pm n_1 a \pm m_1 b) \), encoding the \( (2s + 1) \)-fold topological degeneracy of the NACSL [19]. The gauge factor in Eq. (23) is given by [51]

\[
G(\eta_{n,m}) = (-1)^{m_n+m_r} \exp \left( -i \frac{\Im(b)}{a} (m_\tau - 1 - m) \right) ,
\]

(25)

where \( \Im(b) \) is the real part of \( b \).

The NACSL (23) is approximately annihilated by

\[
\Omega^s_a = \sum_{\beta \neq \alpha} \frac{\vartheta^{\frac{1}{2}}(z|\tau)}{\vartheta^{\frac{1}{2}}(z_a|\tau)} \left( S_{a \beta} \right)^2 ,
\]

(26)

for all \( \alpha \). The prime indicates that we restrict the sum such that the \( (\eta_a - \eta_\beta) \)'s (and not the \( \eta_\beta \)'s) are located in the principal region. In the numerator, we can choose any of the three even Jacobi theta functions: \((u,v) = (0,0), (0,\frac{1}{2}), \) or \((\frac{1}{2},0)\). Note that \( \Omega^s_a [\psi_0^s] \) is not strictly periodic, but only quasiperiodic, due to the shift of the boundary phases inherent in Eq. (26). The statement \( \Omega^s_a [\psi_0^s] \approx 0 \) becomes exact as \( N \to \infty \).

The NACSL (23) is hence the approximate ground state of Eq. (20) [and for \( s = \frac{1}{2} \) also of Eq. (21)] with (13) replaced by

\[
\omega_{a \beta \gamma} = \vartheta^{\frac{1}{2}}(z|\tau) \left( \vartheta^{\frac{1}{2}}(z_a|\tau) \right)^* \vartheta^{\frac{1}{2}}(z_{\beta \gamma}|\tau) \vartheta^{\frac{1}{2}}(z_{\gamma \gamma}|\tau) ,
\]

(27)

where \( * \) denotes complex conjugation, and the sums over \( \beta \) and \( \gamma \) are replaced by primed sums as defined in Eq. (26). As in the case with open boundary conditions, the model becomes exact in the TD limit.

Conclusion. We have identified a parent Hamiltonian for the non-Abelian CSL states [13] which becomes exact in the TD limit. This Hamiltonian should allow us to study the spinon and holon excitations including the non-Abelian braiding properties within a concise framework. The construction also extends to the Abelian \( s = \frac{1}{2} \) Kalmeyer–Laughlin CSL [24,46], where it is likewise exact only as the number of sites \( N \to \infty \), but is considerably simpler that the SKTG Hamiltonian [44,45].

Note added in the proof. After this work was completed, we became aware of a manuscript by Nielsen, Cirac, and Sierra [54], in which they derive the starters grant Topolectrics ERC-StG-Thomale-336012.