Parent Hamiltonian for the non-Abelian chiral spin liquid

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We construct a parent Hamiltonian for the family of non-Abelian chiral spin liquids proposed recently by two of us [Phys. Rev. Lett. **102**, 207203 (2009)] which includes the Abelian chiral spin liquid proposed by Kalmeyer and Laughlin as the special case $s=\frac{1}{2}$. As we use a circular disk geometry with an open boundary, both the annihilation operators we identify and the Hamiltonians we construct from these are exact only in the thermodynamic limit.

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Introduction. The field of two-dimensional quantum spin liquids [1–15] is witnessing a renaissance of interest in present days [16–20]. For one thing, due to advances in the computer facilities available, evidence for spin liquid states in a range of models is accumulating [21,22]. At the same time, spin liquids constitute the most intricate, and in general probably least understood, examples of topological phases [23–28], which themselves establish another vividly studied branch of condensed matter physics [29–31]. If a complete description of the electronic states in the two-dimensional (2D) CuO planes of high T_c superconductors [32] ever emerges, the theory is likely based on a spin $s=1/2$ liquid on a square lattice, which is stabilized through the kinetic energy of itinerant holon excitations [1].

Intimately related to the field of topological phases are the concepts of fractional quantization, and in particular fractional statistics [33]. This field has experienced another seemingly unrelated renaissance of interest in recent years, due to possible applications of states supporting excitations with non-Abelian statistics [34] to the rapidly evolving field of quantum computing and cryptography. The paradigm for this class is the Pfaffian state [35,36], which has been proposed to describe the experimentally observed quantized Hall plateau at Landau level filling fraction $\nu=\frac{5}{2}$ [36]. The state supports quasiparticle excitations which possess Majorana fermion states at zero energy [37]. Braiding of these half vortices yields nontrivial changes in the occupations of the Majorana fermion states, and hence render the exchanges noncommutative or non-Abelian [38,39]. Since this "internal" state vector is insensitive to local perturbations, it is preeminently suited for applications as protected qubits in quantum computation [40,41]. Non-Abelian anyons are further established in other quantum Hall states including Read-Rezayi states [42], in the non-Abelian phase of the Kitaev model [8], the Yao–Kivelson and Yao–Lee models [10,18], and in the family of non-Abelian chiral spin liquid (NACSL) states introduced by two of us [13]. Very recently, non-Abelian statistics has been observed numerically in hard-core lattice bosons in a magnetic field, without reference to explicit wave functions [43].

In this paper, we construct a parent Hamiltonian for the NACSL states [13]. These spin liquids support spinon excitations with SU(2) level $k=2s$ statistics for spin s, i.e., Abelian, Ising, and Fibonacci anyons for $s=\frac{1}{2},1$, and $\frac{3}{2}$, respectively. The method we employ here is different from the method we used to identify a Hamiltonian [44,45] which singles out the Kalmeyer–Laughlin chiral spin liquid (CSL) state [2,46] as its (modulo the twofold topological degeneracy) unique ground state for periodic boundary conditions (PBCs). It is considerably simpler, applicable to the entire family of spin s NACSL states, but exact only in the thermodynamic (TD) limit even if we impose PBCs.

Chiral spin liquid states. The conceptually simplest way to construct the non-Abelian chiral spin liquid (NACSL) state [13] with spin s is to combine $2s$ identical copies of Abelian CSL states with spin $\frac{1}{2}$, and project the spin on each site onto spin s,

$$\underbrace{\tfrac{\mathbf{1}}{\mathbf{2}}\otimes\tfrac{\mathbf{1}}{\mathbf{2}}\otimes\cdots\otimes\tfrac{\mathbf{1}}{\mathbf{2}}}_{2s}=\boldsymbol{s}\oplus(2s-1)\cdot\boldsymbol{s-1}\oplus\cdots$$

The projection onto the completely symmetric representation can be carried out conveniently using Schwinger bosons [7,47]. For a circular droplet with open boundary conditions occupying N sites on a triangular or square lattice, the Abelian CSL state takes the form

$$\begin{aligned}\left|\psi_0^{\rm KL}\right\rangle &= \sum_{\{z_1,\ldots,z_M\}}\psi_0^{\rm KL}(z_1,\ldots,z_M)\,S_{z_1}^+\cdot\dots\cdot S_{z_M}^+\left|\downarrow\downarrow\cdots\downarrow\right\rangle\\ &= \sum_{\substack{\{z_1,\ldots,z_M;\\ w_1,\ldots,w_M\}}}\psi_0^{\rm KL}(z_1,\ldots,z_M)\,a_{z_1}^+\cdots a_{z_M}^\dagger b_{w_1}^+\cdots b_{w_M}^\dagger\left|0\right\rangle\\ &\equiv \Psi_0^{\rm KL}[a^\dagger,b^\dagger]\left|0\right\rangle,\end{aligned}\tag{1}$$

where

$$\psi_0^{\rm KL}[z]=\prod_{i<i}^{M}(z_i-z_j)^2\prod_{i=1}^{M}G(z_i)\,e^{-\frac{1}{4}|z_i|^2}\tag{2}$$

is a bosonic quantum Hall state in the complex "particle" coordinates $z_i\equiv x_i+iy_i$ supplemented by a gauge factor $G(z_i)$, $M=\frac{N}{2}$, $a^\dagger$ and $b^\dagger$ are Schwinger boson creation operators [7,47,48], and the w_k's are those lattice sites which are not occupied by any of the z_i's. In this notation, we can write the spin s state obtained by the projection as

$$\left|\psi_0^s\right\rangle=\left(\Psi_0^{\rm KL}[a^\dagger,b^\dagger]\right)^{2s}\left|0\right\rangle.\tag{3}$$

The lattice may be anisotropic; we have chosen the lattice constants such that the area of the unit cell spanned by the primitive lattice vectors is set to 2π. For a triangular or square lattice with lattice positions given by $\eta_{n,m}=na+mb$, where a

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and b are the primitive lattice vectors in the complex plane and n and m are integers, the gauge phases are simply $G(\eta_{n,m}) = (-1)^{(n+1)(m+1)}$ [46,49].

The NACSL state can alternatively be written as

$$|\psi_0^s\rangle = \sum_{\{z_1,\ldots,z_{sN}\}} \psi_0^s(z_1,\ldots,z_{sN})\, \tilde{S}^+_{z_1}\cdot\ldots\cdot\tilde{S}^+_{z_{sN}}|-s\rangle_N, \quad (4)$$

where $|-s\rangle_N \equiv \otimes_{\alpha=1}^N |s,-s\rangle_\alpha$ is the "vacuum" state in which all the spins are maximally polarized in the negative $\hat{z}$ direction, and $\tilde{S}^+$ are renormalized spin flip operators which satisfy

$$\frac{1}{\sqrt{(2s)!}}(a^\dagger)^n(b^\dagger)^{(2s-n)}|0\rangle = (\tilde{S}^+)^n|s,-s\rangle. \quad (5)$$

In a basis in which S^z is diagonal, we may write

$$\tilde{S}^+ = \frac{1}{s-S^z+1}\, S^+. \quad (6)$$

Note that Eq. (5) implies

$$S^-(\tilde{S}^+)^n|s,-s\rangle = n(\tilde{S}^+)^{n-1}|s,-s\rangle. \quad (7)$$

The wave functions for the spin s state (3) are then effectively given by bosonic Read-Rezayi states [42] for renormalized spin flips,

$$\psi_0^s[z] = \prod_{m=1}^{2s}\left\{ \prod_{\substack{i,j=(m-1)M+1\\ i<j}}^{mM} (z_i - z_j)^2 \right\} \prod_{i=1}^{sN} G(z_i)e^{-\frac{1}{4}|z_i|^2}, \quad (8)$$

which we understand to be completely symmetrized over the "particle" coordinates z_i. For $s = 1$, they take the form of a Moore-Read state [35,36]

$$\psi_0^{s=1}[z] = \mathrm{Pf}\left(\frac{1}{z_i - z_j}\right)\prod_{i<j}^{N}(z_i - z_j)\prod_{i=1}^{sN} G(z_i)e^{-\frac{1}{4}|z_i|^2}. \quad (9)$$

For the considerations below, it is convenient to write the state in the form

$$|\psi_0^s\rangle = \left[\sum_{\{z_1,\ldots,z_M\}} \psi_0^{\mathrm{KL}}(z_1,\ldots,z_M)\, \tilde{S}^+_{z_1}\cdot\ldots\cdot\tilde{S}^+_{z_M}\right]^{2s} |-s\rangle_N. \quad (10)$$

Since the Abelian KL CSL $|\psi_0^{\mathrm{KL}}\rangle$ is an exact spin singlet in the TD limit $N\to\infty$, and is an approximate singlet for finite N, the same holds for the NACSL $|\psi_0^s\rangle$ as well. This follows from the construction of the Schwinger boson projection (3), but can also be verified directly using Perelomov's identity [50,51]. The Abelian and non-Abelian CSL states trivially violate parity (P) and and time reversal (T) symmetry, which would take $z\to\bar{z}$.

Ground state annihilation operators. In the TD limit $N\to\infty$, the NACSL ground states are annihilated by

$$\Omega_\alpha^s = \sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{N} \frac{1}{\eta_\alpha - \eta_\beta}(S_\alpha^-)^{2s}S_\beta^-, \quad \Omega_\alpha^s|\psi_0^s\rangle = 0 \quad \forall\alpha, \quad (11)$$

as we will verify now.

Let us consider the action of $(S_\alpha^-)^{2s}S_\beta^-$ on $|\psi_0^s\rangle$ written in the form (10). Since $\psi_0^{\mathrm{KL}}(z_1,\ldots,z_M)$ vanishes whenever two arguments z_i coincide, one of the z_i's in each of the $2s$ copies in (10) must equal η_α; since $\psi_0^{\mathrm{KL}}(z_1,\ldots,z_M)$ is symmetric under interchange of the z_i's and we count each distinct configuration in the sums over $\{z_1,\ldots,z_M\}$ only once, we may take $z_1=\eta_\alpha$. Regarding the action of S_β^- on (10), we have to distinguish between configurations with $n=0,1,2,\ldots,2s$ renormalized spin flips $\tilde{S}_\beta^+$ at site β. Since the state is symmetric under interchange of the $2s$ copies, we may assume that the n spin flips are present in the first n copies, and account for the restriction through ordering by a combinatorial factor. This yields

$$\begin{aligned}
(S_\alpha^-)^{2s}S_\beta^-|\psi_0^s\rangle &= (S_\alpha^-)^{2s}S_\beta^-\sum_{n=0}^{2s}\binom{2s}{n}\left[\sum_{\{z_3,\ldots,z_M\}}\psi_0^{\mathrm{KL}}(\eta_\alpha,\eta_\beta,z_3,\ldots)\,\tilde{S}^+_\alpha\tilde{S}^+_\beta\tilde{S}^+_{z_3}\cdot\ldots\cdot\tilde{S}^+_{z_M}\right]^n\\
&\quad\times\left[\sum_{\{z_2,\ldots,z_M\}\neq\eta_\beta}\psi_0^{\mathrm{KL}}(\eta_\alpha,z_2,\ldots)\,\tilde{S}^+_\alpha\tilde{S}^+_{z_2}\cdot\ldots\cdot\tilde{S}^+_{z_M}\right]^{2s-n}|-s\rangle_N\\
&= (2s)!\,2s\left[\sum_{\{z_2,\ldots,z_M\}}\psi_0^{\mathrm{KL}}(\eta_\alpha,\eta_\beta,z_3,\ldots,z_M)\,\tilde{S}^+_{z_3}\cdot\ldots\cdot\tilde{S}^+_{z_M}\right]\sum_{n=1}^{2s}\binom{2s-1}{n-1}\\
&\quad\times\left[\sum_{\{z_3,\ldots,z_M\}}\psi_0^{\mathrm{KL}}(\eta_\alpha,\eta_\beta,z_3,\ldots,z_M)\,\tilde{S}^+_\beta\tilde{S}^+_{z_3}\cdot\ldots\cdot\tilde{S}^+_{z_M}\right]^{n-1}\\
&\quad\times\left[\sum_{\{z_2,\ldots,z_M\}\neq\eta_\beta}\psi_0^{\mathrm{KL}}(\eta_\alpha,z_2,\ldots,z_M)\,\tilde{S}^+_{z_2}\cdot\ldots\cdot\tilde{S}^+_{z_M}\right]^{2s-n}|-s\rangle_N
\end{aligned}$$

$$= (2s)!\,2s\left[\sum_{\{z_3,\ldots,z_M\}}\psi_0^{\mathrm{KL}}(\eta_\alpha,\eta_\beta,z_3,\ldots,z_M)\,\tilde S^+_{z_3}\cdot\ldots\cdot\tilde S^+_{z_M}\right]$$

$$\times\left[\sum_{\{z_2,\ldots,z_M\}}\psi_0^{\mathrm{KL}}(\eta_\alpha,z_2,\ldots,z_M)\,\tilde S^+_{z_2}\cdot\ldots\cdot\tilde S^+_{z_M}\right]^{2s-1}|{-s}\rangle_N\,,$$

where we have used Eq. (7). This implies

$$\Omega^s_\alpha\left|\psi^s_0\right\rangle = (2s)!\,2s\left[\sum_{\{z_3,\ldots,z_M\}}\underbrace{\sum_{\beta=1}^{N}\frac{\psi_0^{\mathrm{KL}}(\eta_\alpha,\eta_\beta,z_3,\ldots,z_M)}{\eta_\alpha-\eta_\beta}}_{=0}\tilde S^+_{z_3}\cdot\ldots\cdot\tilde S^+_{z_M}\right]$$

$$\times\left[\sum_{\{z_2,\ldots,z_M\}}\psi_0^{\mathrm{KL}}(\eta_\alpha,z_2,\ldots,z_M)\,\tilde S^+_{z_2}\cdot\ldots\cdot\tilde S^+_{z_M}\right]^{2s-1}|{-s}\rangle_N=0,$$

where we have used the Perelomov identity [50,51] which states that any infinite lattice sum of $e^{-\frac{1}{4}|\eta_\beta|^2}G(\eta_\beta)$ times any analytic function of η_β vanishes.

Parent Hamiltonian. A Hermitian, positive semidefinite, and translationally invariant operator which annihilates $\left|\psi^s_0\right\rangle$ is given by

$$\Gamma\equiv\sum_{\alpha=1}^{N}\Omega^{s\,\dagger}_\alpha\Omega^s_\alpha=\sum_{\substack{\alpha,\beta,\gamma\\ \alpha\neq\beta,\gamma}}\omega_{\alpha\beta\gamma}(S^+_\alpha)^{2s}(S^-_\alpha)^{2s}S^+_\beta S^-_\gamma,\qquad(12)$$

where

$$\omega_{\alpha\beta\gamma}\equiv\frac{1}{\bar\eta_\alpha-\bar\eta_\beta}\frac{1}{\eta_\alpha-\eta_\gamma}.\qquad(13)$$

This operator is not invariant under SU(2) spin rotations, but rather consists of a scalar, vector, and higher tensor components up to order $4s+2$. Since the NACSL states $\left|\psi^s_0\right\rangle$ are spin singlets, and are annihilated by Γ, all these tensor components must annihilate the state individually [52]. The scalar component of Γ, which we denote as $\{\Gamma\}_0$, provides us with an SU(2) spin rotationally invariant parent Hamiltonian.

To obtain the projected operator $\{\Gamma\}_0$, we follow the method described in detail in Ref. [52], and summarize here only the most important steps. With the tensor content of $S^+_\beta S^-_\gamma$ given by

$$S^+_\beta S^-_\gamma=\frac{2}{3}\boldsymbol{S}_\beta\boldsymbol{S}_\gamma-\mathrm{i}(\boldsymbol{S}_\beta\times\boldsymbol{S}_\gamma)^z-\frac{1}{\sqrt6}T^0_{\beta\gamma},\qquad(14)$$

where

$$T^0_{\beta\gamma}=\frac{2}{\sqrt6}\big(3S^z_\beta S^z_\gamma-\boldsymbol{S}_\beta\boldsymbol{S}_\gamma\big)\qquad(15)$$

is the $m=0$ component of the second order tensor, we only need to know the scalar, vector, and second order tensor components of $(S^+_\alpha)^{2s}(S^-_\alpha)^{2s}$ in order to obtain the scalar component of Γ. These are given by (see Sec. 5.3.2 of Ref. [52])

$$(S^+_\alpha)^{2s}(S^-_\alpha)^{2s}=a_0\left\{1+a\,S^z_\alpha+b\,T^0_{\alpha\alpha}+\text{higher orders}\right\}\qquad(16)$$

where

$$a_0=\frac{(2s)!^2}{2s+1},\quad a=\frac{3}{s+1},\quad b=\frac{\sqrt6}{2}\frac{5}{(s+1)(2s+3)}.\qquad(17)$$

The scalar component of Γ is hence given by

$$\{\Gamma\}_0=a_0\sum_{\substack{\alpha,\beta,\gamma\\ \alpha\neq\beta,\gamma}}\omega_{\alpha\beta\gamma}\times\left[\frac{2}{3}\boldsymbol{S}_\beta\boldsymbol{S}_\gamma-\frac{\mathrm{i}a}{3}\boldsymbol{S}_\alpha(\boldsymbol{S}_\beta\times\boldsymbol{S}_\gamma)-\frac{b}{\sqrt6}\left\{T^0_{\alpha\alpha}T^0_{\beta\gamma}\right\}_0\right].\qquad(18)$$

With $\boldsymbol{S}_\beta\times\boldsymbol{S}_\beta=\mathrm{i}\boldsymbol{S}_\beta$ and (see Sec. 4.5.3 of Ref. [52])

$$5\left\{T^0_{\alpha\alpha}T^0_{\beta\gamma}\right\}_0=-\tfrac{4}{3}\boldsymbol{S}^2_\alpha(\boldsymbol{S}_\beta\boldsymbol{S}_\gamma)+2\delta_{\beta\gamma}\boldsymbol{S}_\alpha\boldsymbol{S}_\beta+2[(\boldsymbol{S}_\alpha\boldsymbol{S}_\beta)(\boldsymbol{S}_\alpha\boldsymbol{S}_\gamma)+(\boldsymbol{S}_\alpha\boldsymbol{S}_\gamma)(\boldsymbol{S}_\alpha\boldsymbol{S}_\beta)],\qquad(19)$$

we obtain the final parent Hamiltonian

$$H^s=\sum_{\alpha\neq\beta}\omega_{\alpha\beta\beta}\left[s(s+1)^2+\boldsymbol{S}_\alpha\boldsymbol{S}_\beta-\frac{(\boldsymbol{S}_\alpha\boldsymbol{S}_\beta)^2}{(s+1)}\right]+\sum_{\substack{\alpha,\beta,\gamma\\ \alpha\neq\beta\neq\gamma\neq\alpha}}\omega_{\alpha\beta\gamma}\times\left[(s+1)\boldsymbol{S}_\beta\boldsymbol{S}_\gamma-\frac{2s+3}{2(s+1)}\mathrm{i}\boldsymbol{S}_\alpha(\boldsymbol{S}_\beta\times\boldsymbol{S}_\gamma)-\frac{(\boldsymbol{S}_\alpha\boldsymbol{S}_\beta)(\boldsymbol{S}_\alpha\boldsymbol{S}_\gamma)+(\boldsymbol{S}_\alpha\boldsymbol{S}_\gamma)(\boldsymbol{S}_\alpha\boldsymbol{S}_\beta)}{2(s+1)}\right].\qquad(20)$$

[It is related to Eq. (18) via $\{\Gamma\}_0 = 2a_0/(2s+3)\, H^s$.] This Hamiltonian is approximately valid for any finite disk with N lattice sites, and becomes exact in the TD limit $N \to \infty$, where $H^s|\psi_0^s\rangle = 0$. Note that the $\boldsymbol{S}_\alpha(\boldsymbol{S}_\beta \times \boldsymbol{S}_\gamma)$ term explicitly breaks P and T. (It would be highly desirable to identify a parent Hamiltonian which is P and T invariant, such that the ground states violate these symmetries spontaneously, but we have so far not succeeded in finding one.)

The special case $s = \frac{1}{2}$. Since $S_\alpha^{+2} = 0$ for $s = \frac{1}{2}$, $T_{\alpha\alpha}^m = 0$ for all m, and $\{T_{\alpha\alpha}^0 T_{\beta\gamma}^0\}_0 = 0$. This simplifies Eq. (18) significantly, and yields the parent Hamiltonian

$$H^{s=\frac{1}{2}} = \sum_{\alpha \neq \beta} \omega_{\alpha\beta\beta}\left[\frac{3}{4} + \boldsymbol{S}_\alpha \boldsymbol{S}_\beta\right] + \sum_{\substack{\alpha,\beta,\gamma \\ \alpha \neq \beta \neq \gamma \neq \alpha}} \omega_{\alpha\beta\gamma}[\boldsymbol{S}_\beta \boldsymbol{S}_\gamma - \mathrm{i}\boldsymbol{S}_\alpha(\boldsymbol{S}_\beta \times \boldsymbol{S}_\gamma)]. \tag{21}$$

[It is related to Eq. (18) via $\{\Gamma\}_0 = 2a_0/3\, H^{s=\frac{1}{2}}$.] In contrast to the earlier parent Hamiltonian proposed in Refs. [44] and [45] (SKTG) for the Abelian KL CSL (2) with periodic boundary conditions, Eq. (21) is not exact for finite N. It is considerably simpler than the SKTG model, and, like Eq. (20), becomes exact in the TD limit.

Remarks on periodic boundary conditions. It is rather straightforward to formulate the model on a torus. For simplicity, we choose the lattice constant a real, and b such that the imaginary part $\Im(b) > 0$. We implement PBCs in both directions by identifying the sites z_i, $z_i + L$, and $z_i + L\tau$, where $L = n_1 a$, $L\tau = n_\tau a + m_\tau b$, and $\Im(\tau) > 0$. n_1 and m_τ are positive integers such that the number of sites $N = n_1 m_\tau$ is even, and n_τ is an integer. We place the lattice sites at positions

$$\eta_{n,m} = \left(n - \frac{n_1 - 1}{2}\right)a + \left(m - \frac{m_\tau - 1}{2}\right)b, \tag{22}$$

with $n = 0,1,\ldots,n_1 - 1$ and $m = 0,1,\ldots,m_\tau - 1$. Then the wave function of the NACSL (8) takes the form

$$\psi_0^s[z] = \prod_{m=1}^{2s}\left\{\prod_{\substack{i,j=(m-1)M+1 \\ i<j}}^{mM} \vartheta_{\frac{1}{2},\frac{1}{2}}\left(\tfrac{1}{L}(z_i - z_j)\middle|\tau\right)^2 \times \prod_{v=1}^{2} \vartheta_{\frac{1}{2},\frac{1}{2}}\left(\tfrac{1}{L}(Z_m - Z_{v,m})\middle|\tau\right)\right\} \cdot \prod_{i=1}^{sN} G(z_i) e^{-\frac{1}{2}y_i^2}, \tag{23}$$

where $\vartheta_{\frac{1}{2},\frac{1}{2}}(z|\tau)$ is the odd Jacobi theta function [53], and

$$Z_m \equiv \sum_{i=(m-1)M+1}^{mM} z_i, \quad Z_{1,m} = -Z_{2,m} \tag{24}$$

are the center-of-mass coordinates and zeros, respectively. The latter can be chosen anywhere within the principal region bounded by the four points $\frac{1}{2}(\pm n_1 a \pm m_\tau b)$, encoding the $(2s+1)$-fold topological degeneracy of the NACSL [19]. The gauge factor in Eq. (23) is given by [51]

$$G(\eta_{n,m}) = (-1)^{m_\tau n + m} \exp\left(-i\pi \frac{\Re(b)}{a} m(m_\tau - 1 - m)\right), \tag{25}$$

where $\Re(b)$ is the real part of b.

The NACSL (23) is approximately annihilated by

$$\Omega_\alpha^s = \sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^{N}{}' \frac{\vartheta_{u,v}\left(\frac{1}{L}(\eta_\alpha - \eta_\beta)\middle|\tau\right)}{\vartheta_{\frac{1}{2},\frac{1}{2}}\left(\frac{1}{L}(\eta_\alpha - \eta_\beta)\middle|\tau\right)} (S_\alpha^-)^{2s} S_\beta^- \tag{26}$$

for all α. The prime indicates that we restrict the sum such that the $(\eta_\alpha - \eta_\beta)$'s (and not the η_β's) are located in the principal region. In the numerator, we can choose any of the three even Jacobi theta functions: $(u,v) = (0,0)$, $(0,\frac{1}{2})$, or $(\frac{1}{2},0)$. Note that $\Omega_\alpha^s|\psi_0^s\rangle$ is not strictly periodic, but only quasiperiodic, due to the shift of the boundary phases inherent in Eq. (26). The statement $\Omega_\alpha^s|\psi_0^s\rangle \approx 0$ becomes exact as $N \to \infty$.

The NACSL (23) is hence the approximate ground state of Eq. (20) [and for $s = \frac{1}{2}$ also of Eq. (21)] with (13) replaced by

$$\omega_{\alpha\beta\gamma} = \left(\frac{\vartheta_{u,v}\left(\frac{1}{L}(\eta_\alpha - \eta_\beta)\middle|\tau\right)}{\vartheta_{\frac{1}{2},\frac{1}{2}}\left(\frac{1}{L}(\eta_\alpha - \eta_\beta)\middle|\tau\right)}\right)^* \frac{\vartheta_{u,v}\left(\frac{1}{L}(\eta_\alpha - \eta_\gamma)\middle|\tau\right)}{\vartheta_{\frac{1}{2},\frac{1}{2}}\left(\frac{1}{L}(\eta_\alpha - \eta_\gamma)\middle|\tau\right)}, \tag{27}$$

where $*$ denotes complex conjugation, and the sums over β and γ are replaced by primed sums as defined in Eq. (26). As in the case with open boundary conditions, the model becomes exact in the TD limit.

Conclusion. We have identified a parent Hamiltonian for the non-Abelian CSL states [13] which becomes exact in the TD limit. This Hamiltonian should allow us to study the spinon and holon excitations including the non-Abelian braiding properties within a concise framework. The construction also extends to the Abelian $s = \frac{1}{2}$ Kalmeyer–Laughlin CSL [2,46], where it is likewise exact only as the number of sites $N \to \infty$, but is considerably simpler that the SKTG Hamiltonian [44,45].

Note added in the proof. After this work was completed, we became aware of a manuscript by Nielsen, Cirac, and Sierra [54], in which they derive the $s = \frac{1}{2}$ Hamiltonian (21) using null operators in the conformal correlators of the SU(2) level $k = 1$.

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[1] P. W. Anderson, Science **235**, 1196 (1987).
[2] V. Kalmeyer and R. B. Laughlin, Phys. Rev. Lett. **59**, 2095 (1987).
[3] S. A. Kivelson, D. S. Rokhsar, and J. P. Sethna, Phys. Rev. B **35**, 8865 (1987).
[4] X. G. Wen, F. Wilczek, and A. Zee, Phys. Rev. B **39**, 11413 (1989).
[5] R. Moessner and S. L. Sondhi, Phys. Rev. Lett. **86**, 1881 (2001).
[6] L. Balents, M. P. A. Fisher, and S. M. Girvin, Phys. Rev. B **65**, 224412 (2002).

[7] M. Greiter, J. Low Temp. Phys. **126**, 1029 (2002).
[8] A. Kitaev, Ann. Phys. **321**, 2 (2006).
[9] S. V. Isakov, Y. B. Kim, and A. Paramekanti, Phys. Rev. Lett. **97**, 207204 (2006).
[10] H. Yao and S. A. Kivelson, Phys. Rev. Lett. **99**, 247203 (2007).
[11] S. Dusuel, K. P. Schmidt, J. Vidal, and R. L. Zaffino, Phys. Rev. B **78**, 125102 (2008).
[12] P. A. Lee, Science **321**, 1306 (2008).
[13] M. Greiter and R. Thomale, Phys. Rev. Lett. **102**, 207203 (2009).
[14] M. Hermele, V. Gurarie, and A. M. Rey, Phys. Rev. Lett. **103**, 135301 (2009).
[15] L. Balents, Nature (London) **464**, 199 (2010).
[16] Y. Zhang, T. Grover, and A. Vishwanath, Phys. Rev. B **84**, 075128 (2011).
[17] T. Grover and T. Senthil, Phys. Rev. Lett. **107**, 077203 (2011).
[18] H. Yao and D.-H. Lee, Phys. Rev. Lett. **107**, 087205 (2011).
[19] B. Scharfenberger, R. Thomale, and M. Greiter, Phys. Rev. B **84**, 140404 (2011).
[20] A. E. B. Nielsen, J. I. Cirac, and G. Sierra, J. Stat. Mech.: Theor. Exp. (2011) P11014.
[21] Z. Y. Meng, T. C. Lang, S. Wessel, F. F. Assaad, and A. Muramatsu, Nature (London) **464**, 847 (2010).
[22] H.-C. Jiang, H. Yao, and L. Balents, Phys. Rev. B **86**, 024424 (2012).
[23] X. G. Wen, Phys. Rev. B **40**, 7387 (1989).
[24] X. G. Wen and Q. Niu, Phys. Rev. B **41**, 9377 (1990).
[25] X. Wen, *Quantum Field Theory of Many-Body Systems, Oxford Graduate Texts* (Oxford University Press, New York, 2004).
[26] M. A. Levin and X. G. Wen, Phys. Rev. B **71**, 045110 (2005).
[27] S. B. Chung, H. Yao, T. L. Hughes, and E.-A. Kim, Phys. Rev. B **81**, 060403 (2010).
[28] S. V. Isakov, M. B. Hastings, and R. G. Melko, Nat. Phys. **7**, 772 (2011).
[29] J. E. Moore, Nature (London) **464**, 194 (2010).
[30] M. Z. Hasan and C. L. Kane, Rev. Mod. Phys. **82**, 3045 (2010).
[31] X.-L. Qi and S.-C. Zhang, Rev. Mod. Phys. **83**, 1057 (2011).
[32] J. Zaanen, S. Chakravarty, T. Senthil, P. Anderson, P. Lee, J. Schmalian, M. Imada, D. Pines, M. Randeria, C. Varma, M. Vojta, and M. Rice, Nat. Phys. **2**, 138 (2006).
[33] F. Wilczek, *Fractional Statistics and Anyon Superconductivity* (World Scientific, Singapore, 1990).
[34] A. Stern, Nature (London) **464**, 187 (2010).
[35] G. Moore and N. Read, Nucl. Phys. B **360**, 362 (1991).
[36] M. Greiter, X. G. Wen, and F. Wilczek, Nucl. Phys. B **374**, 567 (1992).
[37] N. Read and D. Green, Phys. Rev. B **61**, 10267 (2000).
[38] D. A. Ivanov, Phys. Rev. Lett. **86**, 268 (2001).
[39] A. Stern, F. von Oppen, and E. Mariani, Phys. Rev. B **70**, 205338 (2004).
[40] A. Y. Kitaev, Ann. Phys. **303**, 2 (2003).
[41] C. Nayak, S. H. Simon, A. Stern, M. Freedman, and S. Das Sarma, Rev. Mod. Phys. **80**, 1083 (2008).
[42] N. Read and E. Rezayi, Phys. Rev. B **59**, 8084 (1999).
[43] E. Kapit, P. Ginsparg, and E. Mueller, Phys. Rev. Lett. **108**, 066802 (2012).
[44] D. F. Schroeter, E. Kapit, R. Thomale, and M. Greiter, Phys. Rev. Lett. **99**, 097202 (2007).
[45] R. Thomale, E. Kapit, D. F. Schroeter, and M. Greiter, Phys. Rev. B **80**, 104406 (2009).
[46] V. Kalmeyer and R. B. Laughlin, Phys. Rev. B **39**, 11879 (1989).
[47] D. P. Arovas, A. Auerbach, and F. D. M. Haldane, Phys. Rev. Lett. **60**, 531 (1988).
[48] J. Schwinger, in *Quantum Theory of Angular Momentum*, edited by L. Biedenharn and H. van Dam (Academic Press, New York, 1965).
[49] Z. Zou, B. Doucot, and B. S. Shastry, Phys. Rev. B **39**, 11424 (1989).
[50] A. M. Perelomov, Theor. Math. Phys. **6**, 156 (1971).
[51] M. Greiter and R. Thomale, Phys. Rev. B **85**, 155145 (2012).
[52] M. Greiter, *Mapping of Parent Hamiltonians*, Vol. 244 of Springer Tracts in Modern Physics (Springer, Berlin/Heidelberg, 2011); arXiv:1109.6104.
[53] D. Mumford, *Tata Lectures on Theta* (Birkhäuser, Basel, 1983), Vols. I and II.
[54] A. E. B. Nielsen, J. I. Cirac, and G. Sierra, Phys. Rev. Lett. **108**, 257206 (2012).